Worksheet 4.6 Sigma Notation

Section 1 Introduction to Sigma Notation

Sigma notation is used as a convenient shorthand notation for the summation of terms.

Example 1 : We write

$$\sum_{n=1}^{5} n = 1 + 2 + 3 + 4 + 5.$$

Here the symbol $\sum$ (sigma) indicates a sum. The numbers at the top and bottom of sigma are called boundaries and tell us what numbers we substitute in to the expression for the terms in our sum. What comes after sigma is an algebraic expression representing terms in the sum. In the example above, n is a variable and represents the terms in our sum.

Example 2 :

$$\sum_{n=1}^{5} n^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3.$$

Example 3 :

$$\sum_{n=3}^{5} n^3 = 3^3 + 4^3 + 5^3.$$

Example 4 :

$$\sum_{n=1}^{4} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$$

Note that we have $\sum_{n=1}^{5} n = \sum_{i=1}^{5} i$. The n and the i just play the role of dummy variables.

We can also work the other way. Sometimes our sum has a pattern which enables us to write the sum using sigma notation.

Example 5 : Write the expression $3 + 6 + 9 + 12 + \cdots + 60$ in sigma notation.

- notice that we are adding multiples of 3;
- so we can write this sum as $\sum_{n=1}^{20} 3n$.

Example 6 : Write the expression $1 + \frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \cdots + \frac{1}{3n+1}$ in sigma notation.

- notice that we are adding fractions with a numerator of 1 and denominators starting with 1 in the first term and then increasing by 3 in each subsequent term;
- i.e. the denominator can be represented by $3k + 1$ for $k = 0, 1, \ldots, n$;
- so we can write this sum as $\sum_{k=0}^{n} \frac{1}{3k+1}$.

We can also use sigma notation when we have variables in our terms.

Example 7 : Write the expression $3x+6x^2+9x^3+12+\cdots+60x^{20}$ in sigma notation.

- note from Example 5 the numbers are multiples of 3 and can be represented by $3n$ where $n = 1, 2, \ldots, 20$;
- we also have powers of x which increase by 1 in each subsequent term;
- so we can write this sum as $\sum_{n=1}^{20} 3nx^n$.

The numbers in front of the variables are called coefficients. In Example 7 the coefficient of x is 3 and the coefficient of x^2 is 6.

Example 8 : Write the expression $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!}$ in sigma notation.

- here the powers of x are even numbers which can be represented by $2k$ for $k = 0, 1, \ldots, n$;
- the denominators are also even numbers but with factorials;
- so we can write this sum as $\sum_{k=0}^{n} \frac{x^{2k}}{(2k)!}$.

Exercises:

1. Write out each of the following sums.

(a) $\displaystyle\sum_{n=1}^{6} n^4$

(b) $\displaystyle\sum_{k=3}^{7} \frac{k+1}{k}$

(c) $\displaystyle\sum_{i=2}^{n} (2i-1)$

(d) $\displaystyle\sum_{k=0}^{n} 2^{k+1}x^k$

(e) $\displaystyle\sum_{k=0}^{n} \frac{(-1)^k x^k}{2k+1}$

2. Express each of these sums using sigma notation.

(a) $1+4+9+16+25+36$

(b) $3-5+7-9+11-13+15$

(c) $\frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\frac{1}{14}+\frac{1}{17}$

(d) $\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\cdots+\frac{n+1}{n+2}$

(e) $2-2^2+2^3-2^4+\cdots+2^{2n+1}$

(f) $2x^3+4x^5+6x^7+\cdots+30x^{31}$

(g) $x-x^2+\frac{x^3}{2!}-\frac{x^4}{3!}+\frac{x^5}{4!}-\frac{x^6}{5!}$

(h) $3x+7x^2+11x^3+15x^4+19x^5+23x^6$

(i) $8x^4+10x^5+12x^6+\cdots+(2n+2)x^{n+1}$

(j) $12x^4+20x^5+30x^6+\cdots+n(n-1)x^n$

(k) $\frac{x^3}{2}-\frac{x^5}{3}+\frac{x^7}{4}-\frac{x^9}{5}+\cdots-\frac{x^{199}}{100}$

Section 2 Finding Coefficients

Sigma notation is a useful way to express the sum of a large number of terms. When we want to find particular terms or coefficients, we don't always have to expand the whole expression to find it.

Example 1 : Find the coefficient of x^4 in $\displaystyle\sum_{k=0}^{8}(4k+3)x^k$.

- the terms in this sum look like $(4k+3)x^k$;
- the terms with x^4 occurs when $k=4$ i.e. $(4(4)+3)\,x^4 = 19x^4$;
- the coefficient of x^4 is 19.

Example 2 : Find the coefficient of x^7 in $\displaystyle\sum_{k=0}^{8}(4k+3)x^{k+2}$.

- a typical term is of the form $(4k+3)x^{k+2}$;

- the term with x^7 occurs when $k+2=7$ i.e. $k=5$;
- we have $(4(5)+3)\,x^{5+2} = 23x^7$;
- the coefficient of x^7 is 23.

Example 3 : Find the coefficient of x^2 in $(3+x)\sum_{k=0}^{8}(4k+3)x^k$.

- we can think of this as $3\sum_{k=0}^{8}(4k+3)x^k + x\sum_{k=0}^{8}(4k+3)x^k$;
- the term with x^2 can be obtained by taking $k=2$ from the first part of this expression to get $3\,(4(2)+3)\,x^2 = 33x^2$ and then taking $k=1$ from the second part of this expression to get $x\,(4(1)+3)\,x^1 = 7x^2$;
- combining these we get $33x^2 + 7x^2 = 40x^2$;
- so the coefficient of x^2 is 40.

Exercises:

1. Find the coefficients of x^2 and x^6 in the following.

   (a) $\displaystyle\sum_{r=0}^{10}\frac{r+1}{r!}x^r$

   (b) $\displaystyle\sum_{k=3}^{15}k(k+1)x^{k-2}$

   (c) $\displaystyle\sum_{n=0}^{20}\frac{(-1)^n x^{4n+2}}{n+3}$

   (d) $\displaystyle(3+2x)\sum_{k=0}^{8}(k+1)x^k$

   (e) $\displaystyle(1-x)\sum_{k=0}^{7}\frac{x^{k+1}}{k!}$

   (f) $\displaystyle(x+x^2)\sum_{k=0}^{15}(2k+1)x^k$

Exercises 4.6 Sigma Notation

1. Write out each of the following sums.

   (a) $\displaystyle\sum_{r=0}^{7} r^2(-x)^r$ (b) $\displaystyle\sum_{k=3}^{8} \frac{k-1}{k+1}x^{2k}$ (c) $\displaystyle\sum_{k=1}^{n+1} k(k-1)x^{3k}$

2. Write each of the following series in sigma notation.

   (a) $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36}$
   (b) $7 - 10 + 13 - 16 + \cdots + 31$
   (c) $4^2 + 5^2 + 6^2 + 7^2 + \cdots + (n+2)^2$
   (d) $\frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} + \cdots + \frac{1}{(n+5)!}$
   (e) $3x^2 + 6x^4 + 9x^6 + 12x^8 + \cdots + 36x^{24}$
   (f) $x^7 + \frac{x^9}{1!} + \frac{x^{11}}{2!} + \frac{x^{13}}{3!} + \cdots + \frac{x^{31}}{12!}$
   (g) $x - 5x^2 + 9x^3 - 13x^4 + \cdots - 41x^{10}$
   (h) $\frac{5}{2}x^3 + 3x^4 + \frac{7}{2}x^5 + 4x^6 + \cdots + \frac{n}{2}x^{n-2}$
   (i) $6x^{12} + 7x^{14} + 8x^{16} + 9x^{18} + \cdots + (n+1)x^{2n+2}$

3. Find the coefficient of x, x^3, and x^7 in the following expressions.

   (a) $\displaystyle\sum_{k=3}^{n} \frac{(-1)^k x^{k-2}}{(4k-1)!}$
   (b) $\displaystyle\sum_{k=1}^{n} \frac{(3x)^{k-3}}{(k+1)^2}$
   (c) $\displaystyle(2+x)\sum_{k=0}^{n} \frac{k-1}{k+1}x^k$
   (d) $\displaystyle(x-x^2)\sum_{k=0}^{n} \frac{(-1)^{k+1}}{3k!}x^k$
   (e) $\displaystyle(5+x^2)\sum_{k=1}^{n} \frac{k}{(2k+1)!}x^{k-2}$

4. Simplify the following expressions.

   (a) $\displaystyle\sum_{k=0}^{n} k^2 - (k+1)^2$ (b) $\displaystyle\sum_{k=1}^{n} \left(\frac{1}{k+1} - \frac{1}{k}\right)$

Answers 4.6

Section 1

1. (a) $1^4 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4$
   (b) $\frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \frac{7}{6} + \frac{8}{7}$
   (c) $3 + 5 + 7 + 9 + \cdots + (2n - 1)$
   (d) $2 + 4x + 6x^2 + 8x^3 + \cdots + 2^{n+1}x^n$
   (e) $1 - \frac{x}{3} + \frac{x^2}{5} - \frac{x^3}{7} + \cdots + \frac{(-1)^n x^n}{2n+1}$

2. (a) $\sum_{n=1}^{6} n^2$
   (b) $\sum_{n=2}^{8} (-1)^n (2n - 1)$
   (c) $\sum_{n=0}^{5} \frac{1}{3n + 2}$
   (d) $\sum_{k=1}^{n} \frac{k + 1}{k + 2}$
   (e) $\sum_{k=1}^{2n+1} (-1)^{k+1} 2^k$
   (f) $\sum_{n=1}^{15} 2nx^{2n+1}$
   (g) $\sum_{n=0}^{5} (-1)^n \frac{x^{n+1}}{n!}$
   (h) $\sum_{n=0}^{5} (4n + 3)x^{n+1}$
   (i) $\sum_{k=3}^{n} (2k + 2)x^{k+1}$
   (j) $\sum_{k=4}^{n} k(k - 1)x^k$
   (k) $\sum_{n=1}^{99} \frac{(-1)^{n+1} x^{2n+1}}{n + 1}$

Section 2

1. (a) $x^2 : \frac{3}{2}, \quad x^6 : \frac{7}{6!}$
   (b) $x^2 : 20, \quad x^6 : 72$
   (c) $x^2 : \frac{1}{3}, \quad x^6 : -\frac{1}{4}$
   (d) $x^2 : 13, \quad x^6 : 33$
   (e) $x^2 : 0, \quad x^6 : \frac{1}{5!} - \frac{1}{4!}$
   (f) $x^2 : 4, \quad x^6 : 20$

Exercises 4.6

1. (a) $-x + 4x^2 - 9x^3 + 16x^4 - 25x^5 + 36x^6 - 49x^7$
   (b) $\frac{2}{4}x^6 + \frac{3}{5}x^8 + \frac{4}{6}x^{10} + \frac{5}{7}x^{12} + \frac{6}{8}x^{14} + \frac{7}{9}x^{16}$
   (c) $2x^6 + 6x^9 + 12x^{12} + 20x^{15} + \cdots + n(n + 1)x^{3n+3}$

2. (a) $\displaystyle\sum_{n=1}^{6} \frac{(-1)^{n+1}}{n^2}$

(b) $\displaystyle\sum_{n=0}^{8} (-1)^n(3n+7)$

(c) $\displaystyle\sum_{k=2}^{n} (k+2)^2$

(d) $\displaystyle\sum_{k=2}^{n} \frac{1}{(k+5)!}$

(e) $\displaystyle\sum_{n=1}^{12} 3nx^{2n}$

(f) $\displaystyle\sum_{n=0}^{12} \frac{x^{2n+7}}{n!}$

(g) $\displaystyle\sum_{k=1}^{10} (-1)^{k+1}(4k+1)x^k$

(h) $\displaystyle\sum_{k=5}^{n} \frac{kx^{k-2}}{2}$

(i) $\displaystyle\sum_{k=6}^{n+1} nx^{2n}$

3. (a) $x : -\frac{1}{11!}, \quad x^3 : -\frac{1}{19!}, \quad x^7 : -\frac{1}{35!}$

(b) $x : \frac{3}{25}, \quad x^3 : \frac{27}{49}, \quad x^7 : \frac{3^7}{121}$

(c) $x : -1, \quad x^3 : \frac{4}{3}, \quad x^7 : \frac{31}{14}$

(d) $x : -\frac{1}{3}, \quad x^3 : -\frac{1}{2}, \quad x^7 : -\frac{1}{3}\left(\frac{1}{6!} + \frac{1}{5!}\right)$

(e) $\frac{15}{7!} + \frac{1}{3!}, \quad x^3 : \frac{25}{11!} + \frac{3}{7!} \quad x^7 : \frac{45}{19!} + \frac{7}{15!}$

4. (a) $-(n+1)^2$

(b) $-\dfrac{n}{n+1}$