THE COMPREHENSIVE FACTORIZATION AND TORSORS

ROSS STREET AND DOMINIC VERITY

Dedicated to Dominique Bourn on the occasion of his 60th birthday.

Abstract. This is an expanded, revised and corrected version of the first author's preprint [1]. The discussion of one-dimensional cohomology H^1 in a fairly general category $\mathcal{E}$ involves passing to the (2-)category $\mathrm{Cat}(\mathcal{E})$ of categories in $\mathcal{E}$. In particular, the coefficient object is a category B in $\mathcal{E}$ and the torsors that H^1classifies are particular functors in $\mathcal{E}$. We only impose conditions on $\mathcal{E}$ that are satisfied also by $\mathrm{Cat}(\mathcal{E})$ and argue that H^1 for $\mathrm{Cat}(\mathcal{E})$ is a kind of H^2 for $\mathcal{E}$, and so on recursively. For us, it is too much to ask $\mathcal{E}$ to be a topos (or even internally complete) since, even if $\mathcal{E}$ is, $\mathrm{Cat}(\mathcal{E})$ is not. With this motivation, we are led to examine morphisms in $\mathcal{E}$ which act as internal families and to internalize the comprehensive factorization of functors into a final functor followed by a discrete fibration. We define B-torsors for a category B in $\mathcal{E}$ and prove clutching and classification theorems. The former theorem clutches Čech cocycles to construct torsors while the latter constructs a coefficient category to classify structures locally isomorphic to members of a given internal family of structures. We conclude with applications to examples.

1. Introduction

For an abelian group B in a Grothendieck topos $\mathcal{E}$, there are two constructions for the abelian cohomology groups $H^n(\mathcal{E}; B)$. The first of these uses the existence of enough injective abelian groups in $\mathcal{E}$ and will not be considered here. The second is a modification of Čech cohomology involving a certain colimit over the directed set of hypercovers of the terminal object 1 in $\mathcal{E}$. We have

$$H^0(\mathcal{E}; B) \cong \mathcal{E}(1, B),$$

the abelian group of global sections of B. In the calculation of $H^1(\mathcal{E}; B)$, we only need to consider covers in place of the more elaborate hypercovers: a cover in $\mathcal{E}$ is an epimorphism $R \twoheadrightarrow 1$.

With suitable choice of $\mathcal{E}$ the above cohomology amounts to the usual cohomology of a space with coefficients in a sheaf B of abelian groups over the space, the usual cohomology of a group with coefficients in a module B over the integral group-ring of the group, etale cohomology, and so on.

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Giraud [2] generalized the constructions $H^n(\mathcal{E}; B)$ for $n = 0, 1, 2$ to accommodate a group B which is not necessarily abelian. There is no difficulty with $n = 0$ where we take the group $\mathcal{E}(1, B)$. For $n = 1$, the notion of B-torsor is required: this is an object E of $\mathcal{E}$ on which B acts such that E is locally isomorphic to B acting on itself ("locally" here means "on passing to a cover"). Then $H^1(\mathcal{E}; B)$ is the pointed set of isomorphism classes of B-torsors. A basic result of Giraud is that, given an object X of $\mathcal{E}$ enriched with some structure, if B denotes the group of structure automorphisms of X, then $H^1(\mathcal{E}; B)$ classifies the objects of $\mathcal{E}$ locally structure isomorphic to X.

The definitions of Giraud can be generalized to an elementary topos $\mathcal{E}$ ([3] and [4]).

The purpose of the present paper is to present a generalization for $n = 1$ to the case where $\mathcal{E}$ is a finitely complete, finitely cocomplete, cartesian closed category and B is a category in $\mathcal{E}$. There have been two basic sources of inspiration for this work. One is the observation of Roberts [5] that what one needs in order to be able to express the n-cocycle condition is an n-category (any set is a 0-category, a group gives a 1-category with one object, an abelian group gives an n-category with one $(n-1)$-cell for all n). This suggests that there should be an n-category of n-cocycles with coefficients in an n-category; 0-cohomology would then be the set of 0-cocycles, 1-cohomology would be the set of isomorphism classes of 1-cocycles, 2-cohomology would be the set of equivalence classes of 2-cocycles, and so on.

André Joyal's lectures in the Category Seminar at Macquarie University (22 October and 5 November 1980) were the other source of inspiration. André stressed the important case where B is a groupoid in a fairly general category $\mathcal{E}$ (a groupoid is a category in which every morphism is invertible; a group is a groupoid with one object). He defined a B-torsor to be a discrete fibration E over B which is locally representable and said that the relationship between B-torsors and cocycles could be explained in terms of the comprehensive factorization of a functor into a final functor followed by a discrete fibration [6].

In a topos it is rewarding to view a morphism $u \colon L \twoheadrightarrow K$ as an internal expression of a family of objects parametrized by K. In a merely finitely complete category in general this point of view makes little sense, so in Section 2 we have suggested the notion of *powerful morphisms* should play this role. Every morphism in a topos is powerful. The powerful morphisms in the category of categories are known (originally due to [11] and [12]; also see Theorem 2.37 of [3]) and include both fibrations and opfibrations. Via the inverse of the Grothendieck construction (see Section 1 of [15] for example) a fibration certainly represents a family of categories parametrized by a category.

In an arbitrary finitely complete and finitely cocomplete category we provide, in Section 2, an efficient proof that each functor into an *amenable category* factors comprehensively and that this factorization is functorial. A category B is amenable when the codomain morphism $d_1 \colon B_1 \twoheadrightarrow B_0$ is powerful. The case of internal groupoids was developed by Dominique Bourn [7].

Section 3 brings us to the notion of a *B-torsor* where B is a category in a finitely complete category $\mathcal{E}$. In topos theory it has been suggested that (in view of Section 8.3

of Johnstone [3]) a B-torsor in a topos $\mathcal{E}$ should perhaps be a *flat* discrete fibration E over B. However, we know of no results concerning such torsors with B not a groupoid except that the topos $\mathcal{E}^B$ would then be the B-torsor classifier. Instead, the definition taken here is that a B-torsor is a discrete fibration over B which is locally representable (that is, on passing to a cover, looks like some $d_0\colon (B/x) \to B$). This leads to a category $\mathrm{Tors}(B)$ of B-torsors. We contend that one-dimensional cohomology is the study of the categories $\mathrm{Tors}(B)$.

It will be clear (after Theorem 4.9) that a B-torsor amounts precisely to a B_g-torsor where B_g is the groupoid obtained from B by restricting to invertible morphisms in B. So one may ask: is it not sufficient to consider B-torsors where B is a groupoid? No! For what we in fact have is an equivalence of categories

$$\mathrm{Tors}(B)_g \simeq \mathrm{Tors}(B_g), \tag{1}$$

and it is not possible to glean all the important information about a category $\mathrm{Tors}(B)$ from the associated groupoid. For example, in Section 6 we give a technique for deriving results about finiteness in a topos, vector bundles, and other local structures, which could not be obtained by restricting to groupoids.

Here, a *cover* is taken to mean an object R of $\mathcal{E}$ for which $R \to 1$ is a regular epimorphism (that is, a coequalizer of some pair of morphisms). When $\mathcal{E}$ is cartesian closed, covers are closed under finite products. All our results apply equally well when covers are replaced by any subclass closed under finite products.

Apart from some basics about powerful morphisms and an internalization of the comprehensive factorization [6] the two main results of this paper are the Clutching Theorem 4.11 and the Classification Theorem 5.11. The first of these establishes the relationship between B-torsors and Čech B-cocycles when B is an amenable category in $\mathcal{E}$. The second, which warrants some explanation here, generalizes the result of Giraud mentioned near the beginning of this Introduction.

When viewing a category $\mathcal{E}$ as a universe of discourse and categories in $\mathcal{E}$ as "small categories", a useful concept of "large category" is that of homomorphism $X\colon \mathcal{E}^{\mathrm{op}} \to \mathrm{Cat}$ (or "pseudofunctor", or, equivalently, fibration over $\mathcal{E}$); for K in $\mathcal{E}$, we think of the objects of XK as "K-indexed families of objects of the large category X". In accordance with our view of powerful morphisms, $\mathcal{E}$ becomes the underlying external category for the large category P with PK taken to be the category of powerful morphisms into K. Categories constructed as consisting of structures in $\mathcal{E}$ (for example, the category of rings in $\mathcal{E}$) also underlie large categories (this means more than the observation that the category of rings in $\mathcal{E}$ has homs enriched in $\mathcal{E}$). The Classification Theorem concerns suitably cocomplete large categories X in $\mathcal{E}$ and K-indexed families x of X-objects; it states that *X-objects locally isomorphic to some member of the family x are classified by $X[x]$-torsors, where $X[x]$ is the small category in $\mathcal{E}$ "of members of the family x and all X-arrows between them."*

Finally, note that, if $\mathcal{C}$ is a finitely complete, finitely cocomplete, cartesian closed category then so too is the category $\mathcal{E}$ underlying the 2-category $\mathrm{Cat}(\mathcal{C})$ of categories in $\mathcal{C}$. Thus by recursion, we can apply our results to multiple categories [15] in a topos: wherein it can be seen (using Theorem 2.16) that n-categories are amenable. For example, an abelian group G in a topos $\mathcal{C}$ gives rise to a group B in $\mathcal{E}$ (as above); with the notion of cover in $\mathcal{E}$ taken to be that of connected groupoid, B-torsors in $\mathcal{E}$ are closely related to elements of $H^2(\mathcal{C}; G)$.

To realise our program of obtaining higher cohomologies as one-dimensional cohomologies under this kind of recursion, we must first further generalise the techniques presented here to a *quasi-categorical* setting. In that context our category $\mathcal{E}$ would come equipped with a homotopy theory, usually given as a Quillen model structure. Furthermore $\mathrm{Cat}(\mathcal{E})$ would be replaced by the category of simplicial objects in $\mathcal{E}$ along with a derived homotopy theory whose fibrant objects are certain kinds of *internal quasi-categories* (or equivalently *complete Segal spaces*). Much of the theory presented here then generalises to that kind of setting, while retaining the conceptual structure presented here.

So why have we not gone straight to describing that generalisation here? Our answer to this question comes in two parts. Firstly, the results we present here already have important applications which cannot be derived from classical presentations of this material. Secondly, in this work we have tried to present the overall structure of our approach to this topic without confusing the reader (or indeed ourselves) with the technical detail involved in its homotopy theoretic generalisation. This enables us to present a simplified conceptual road map which we intend to build upon elsewhere.

2. Powerful morphisms

Considerable use will be made of the Adjoint Triangle Theorem of Dubuc [25] in the following form.

2.1. Lemma. *Suppose the functor $F\colon \mathcal{B} \to \mathcal{X}$ has a right adjoint U with the all components of the unit $\eta\colon 1 \to UF$ regular monomorphisms in $\mathcal{B}$. Any functor $S\colon \mathcal{A} \to \mathcal{B}$ has a right adjoint if and only if the composite FS does.*

Proof. Suppose FS has right adjoint Q. Each morphism $\eta B\colon B \to UFB$ is an equalizer of two morphism into some object C of $\mathcal{B}$. But ηC is a monomorphism. So B is an equalizer of two morphisms $UFB \to UFC$ and it follows that $\mathcal{B}(SA, B)$ is the equalizer of two morphisms $\mathcal{B}(SA, UFB) \to \mathcal{B}(SA, UFC)$, naturally in A. Using $F \dashv U$, we see that $\mathcal{B}(SA, B)$ is the equalizer of two morphisms $\mathcal{X}(FSA, FB) \to \mathcal{X}(FSA, FC)$, naturally in A. Using $FS \dashv Q$, we see that $\mathcal{B}(SA, B)$ is the equalizer of two morphisms $\mathcal{A}(A, QFB) \to \mathcal{A}(A, QFC)$, naturally in A. By Yoneda's Lemma, we obtain two morphisms $QFB \to QFC$ whose equalizer TB has $\mathcal{A}(A, TB) \cong \mathcal{B}(SA, B)$, naturally in A. ∎

2.2. Remark. The assumptions in the first sentence of the lemma hold if F is comonadic. On the other hand, the assumptions on F generally imply that it is conservative (that is reflects invertibility). A related observation is that restriction along (pre-composition by) U reflects the invertibility of natural transformations $\alpha\colon H \Rightarrow K\colon \mathcal{A} \to \mathcal{B}$ between equalizer-preserving functors H and K.

2.3. Definition. *Let $\mathcal{E}$ denote a finitely complete category. A morphism $p\colon E \to B$ in $\mathcal{E}$ is called* ***powerful*** *when the functor*

$$- \underset{B}{\times} E\colon \mathcal{E}_{/B} \longrightarrow \mathcal{E}, \tag{2}$$

which assigns to each object of the slice category $\mathcal{E}_{/B}$ its pullback with p, has a right adjoint. (The term "exponentiable" instead of "powerful" has been used in the literature.)

The following result is adapted to this terminology from Freyd [8].

2.4. Proposition. *The following conditions on a morphism $p\colon E \to B$ in $\mathcal{E}$ are equivalent:*

(1) the morphism p is powerful;

(2) the functor

$$p^*\colon \mathcal{E}_{/B} \longrightarrow \mathcal{E}_{/E}, \tag{3}$$

defined by pulling back along p, has a right adjoint Π_p;

(3) exponentiation exists with respect to the index $p\colon E \to B$ in the category $\mathcal{E}_{/B}$.

Proof. Consider the functor $\Sigma_E\colon \mathcal{E}_{/E} \to \mathcal{E}$ taking each morphism into E to its domain. It has a right adjoint $\Sigma_E \dashv \Delta_E$ taking X to $X \times E$ over E via the second projection. The unit of the adjunction is a componentwise regular monomorphism. By Lemma 2.1, the functor $p^*\colon \mathcal{E}_{/B} \to \mathcal{E}_{/E}$ has a right adjoint if and only if $- \underset{B}{\times} E = \Sigma_E\, p^*$ does. This proves the equivalence of (1) and (2).

Product with the object p of $\mathcal{E}_{/B}$ is the composite

$$\mathcal{E}_{/B} \xrightarrow{\;p^*\;} \mathcal{E}_{/E} \xrightarrow{\;\Sigma_p\;} \mathcal{E}_{/B}$$

where $\Sigma_p \dashv p^*$. It follows that (2) implies (3).

To see that (3) implies (1), notice that the functor $- \underset{B}{\times} E$ is the composite

$$\mathcal{E}_{/B} \xrightarrow{\;p^*\;} \mathcal{E}_{/E} \xrightarrow{\;\Sigma_p\;} \mathcal{E}_{/B} \xrightarrow{\;\Sigma_B\;} \mathcal{E}.$$

∎

2.5. Definition. *In a category $\mathcal{C}$ with finite products, an object P is said to be* ***powerful*** *when the functor $P \times -\colon \mathcal{C} \to \mathcal{C}$ has a right adjoint.*

So, by Proposition 2.4(3), a morphism $p\colon E \to B$ in $\mathcal{E}$ is powerful if and only if it is powerful as an object of the category $\mathcal{E}_{/B}$. An object E of $\mathcal{E}$ is powerful if and only if the morphism $E \to 1$ is powerful.

2.6. Corollary.

(1) *Isomorphisms are powerful.*

(2) *The composite of powerful morphisms is powerful.*

(3) *The pullback of a powerful morphism is powerful.*

Proof. The only one of these that needs any comment is (3) and this follows from Lemma 2.1 since the unit of each adjunction $\Sigma_f \dashv f^*$ is a regular monomorphism. ∎

Given a functor $F\colon \mathcal{E} \to \mathcal{X}$, we write $F_B\colon \mathcal{E}_{/B} \to \mathcal{X}_{/FB}$ for the functor induced by F in the obvious way on slice categories.

2.7. Definition. *For a powerful morphism $p\colon E \to B$, a pullback preserving functor $F\colon \mathcal{E} \to \mathcal{X}$ is said to* ***preserve internal p-products*** *when Fp is powerful and the canonical morphism*

$$F_B \Pi_p \longrightarrow \Pi_{Fp} F_E, \tag{4}$$

obtained as the mate of the inverse of the canonical isomorphism

$$F_E\, p^* \longrightarrow (Fp)^*\, F_B, \tag{5}$$

is invertible.

2.8. Example.

(1) When $\mathcal{E}$ is a category of sets, or more generally a topos, every morphism is powerful. A category in which each morphism is powerful is called *internally complete* (or "locally cartesian closed" [9] or "a closed span category" [10]).

(2) When $\mathcal{E}$ is the category Cat of categories, the powerful morphisms have been identified by Giraud [11] and Conduché [12] (also see [3] and [13]). Moreover, the simplicial nerve functor $N\colon$ Cat $\to \hat{\Delta}$ preserves internal p-products for all powerful functors p (we leave this as an exercise using the references; however, see Proposition 2.10).

2.9. Proposition. *Suppose $S \dashv T\colon \mathcal{E} \to \mathcal{X}$ is an adjunction between finitely complete categories and let $\varepsilon\colon ST \Rightarrow 1$ denote the counit. Suppose $p\colon E \to B$ in $\mathcal{E}$ and $Tp\colon TE \to TB$ in $\mathcal{X}$ are powerful. If S preserves pullbacks along Tp and the following naturality square*

$$\begin{tikzcd} STE \arrow[r, "STp"] \arrow[d, "\varepsilon_E"'] \arrow[dr, phantom, "="] & STB \arrow[d, "\varepsilon_B"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

is a pullback then T preserves internal p-products.

Proof. We are required to prove that $T_B\Pi_p \longrightarrow \Pi_{Tp}T_E$ is invertible. That is, we must show that, for all $f\colon X \to TB$ and $g\colon Y \to E$, the canonical

$$\mathcal{X}_{/TB}(X \xrightarrow{f} TB,\, T_B\Pi_p(Y \xrightarrow{g} E)) \longrightarrow \mathcal{X}_{/TB}(X \xrightarrow{f} TB,\, \Pi_{Tp}T_E(Y \xrightarrow{g} E)) \tag{6}$$

is a bijection. Put $\Pi_p(Y \xrightarrow{g} E) = (V \xrightarrow{h} B)$ and form the following pullback.

$$\begin{tikzcd} P \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "="] & X \arrow[d, "f"] \\ TE \arrow[r, "Tp"'] & TB \end{tikzcd}$$

Using our hypotheses on S and ε, it follows that $SP \xrightarrow{Su} STE \xrightarrow{\varepsilon_E} E$ is the pullback of $SX \xrightarrow{Sf} STB \xrightarrow{\varepsilon_B} B$ along p. So the function (6) decomposes as the composite of bijections:

$$\mathcal{X}_{/TB}(X \xrightarrow{f} TB,\, TV \xrightarrow{Th} TB) \cong \mathcal{E}_{/B}(SX \xrightarrow{Sf} STB \xrightarrow{\varepsilon_B} B,\, V \xrightarrow{h} B) \cong$$
$$\mathcal{E}_{/E}(SP \xrightarrow{Su} STE \xrightarrow{\varepsilon_E} E,\, Y \xrightarrow{g} E) \cong \mathcal{X}_{/TE}(P \xrightarrow{u} TE,\, TY \xrightarrow{Tg} TE) \cong$$
$$\mathcal{X}_{/TB}(X \xrightarrow{f} TB,\, \Pi_{Tp}T_E(Y \xrightarrow{g} E)).$$

■

Choose a category Set of sets containing $\mathcal{E}$ as an internal category. We write $\hat{\mathcal{E}}$ for the topos of Set-valued presheaves on $\mathcal{E}$.

2.10. Proposition. *Suppose that $\mathcal{E}$ and $\mathcal{X}$ are finitely complete categories and that $F\colon \mathcal{E} \to \mathcal{X}$ is a dense, fully faithful, left exact functor. Assume further that $p\colon E \to B$ is a powerful arrow in $\mathcal{E}$ and that $Fp\colon FE \to FB$ is a powerful arrow in $\mathcal{X}$. Then F preserves internal p-products.*

Proof. Start by considering the sliced functor $F_B\colon \mathcal{E}_{/B} \to \mathcal{X}_{/FB}$ for any object $B \in \mathcal{E}$ and observe that:

- F_B is left exact. To prove this, simply consider the commutative square of functors

$$\begin{tikzcd} \mathcal{E}_{/B} \arrow[r, "F_B"] \arrow[d, "\pi_B"'] & \mathcal{X}_{/FB} \arrow[d, "\pi_{FB}"] \\ \mathcal{E} \arrow[r, "F"'] & \mathcal{X} \end{tikzcd}$$

  in which the vertical functors are the canonical projections, which preserve and reflect all finite limits. So we may infer from the assumption that F preserves finite limits that F_B also preserves all finite limits as required.

- F_B is fully faithful. Consider the following, serially commutative, diagram

$$\begin{tikzcd}
\mathcal{E}_{/B}(f\colon E \to B, f'\colon E' \to B) \arrow[r, hook] \arrow[d, "F_B"'] & \mathcal{E}(E,E') \arrow[r, shift left, "\ulcorner f \urcorner \circ !"] \arrow[r, shift right, "\mathcal{E}(E,f')"'] \arrow[d, "\cong"', "F"] & \mathcal{E}(E,B) \arrow[d, "F"', "\cong"] \\
\mathcal{X}_{/FB}(F_B(f), F_B(f')) \arrow[r, hook] & \mathcal{X}(FE,FE') \arrow[r, shift left, "\ulcorner F_B(f) \urcorner \circ !"] \arrow[r, shift right, "\mathcal{X}(FE, F_B(f'))"'] & \mathcal{X}(FE,FB)
\end{tikzcd}$$

  in which the horizontal forks are the equalisers used to define the homsets of the slice categories $\mathcal{E}_{/B}$ and $\mathcal{X}_{/FB}$. Here the right hand and central verticals are both isomorphisms since F is fully faithful. So it follows that the left hand vertical is also an isomorphism, since pointwise isomorphisms of diagrams induce isomorphisms of limits, as required.

- F_B is dense. Of course, a functor $F\colon \mathcal{E} \to \mathcal{X}$ is dense if and only if the associated functor

$$\begin{aligned} \tilde{F}\colon \mathcal{X} &\longrightarrow \hat{\mathcal{E}} \\ X &\longmapsto \mathcal{X}(F(-), X) \end{aligned}$$

  is fully faithful. So to show that $F_B\colon \mathcal{E}_{/B} \to \mathcal{X}_{/FB}$ is dense we must show that its associated functor

$$\widetilde{F_B}\colon \mathcal{X}_{/FB} \longrightarrow \widehat{\mathcal{E}_{/B}}$$

  is fully faithful. To do this, consider first the sliced functor:

$$\tilde{F}_{FB}\colon \mathcal{X}_{/FB} \longrightarrow \hat{\mathcal{E}}_{/\tilde{F}FB}$$

  From the assumption of the density of F we may infer that the associated functor $\tilde{F}$ is fully faithful, from which it follows, by the argument of the last point of this proof, that our sliced functor $\tilde{F}_{FB}$ is also fully faithful. Now observe that

$$\begin{aligned} \tilde{F}FB(B') &= \mathcal{X}(FB', FB) && \text{by definition of } \tilde{F} \\ &\cong \mathcal{E}(B', B) && \text{since } F \text{ fully faithful} \end{aligned}$$

  so $\tilde{F}FB \cong \mathcal{Y}_{\mathcal{E}}B$. Now we may apply the fact that a slice $\hat{\mathcal{E}}_{/X}$ of a presheaf category is itself equivalent to the presheaf category $\widehat{\mathbb{G}(X)}$ on the Grothendieck category constructed from X, to show that:

$$\begin{aligned} \hat{\mathcal{E}}_{/\tilde{F}FB} &\simeq \widehat{\mathbb{G}(\tilde{F}FB)} \\ &\cong \widehat{\mathbb{G}(\mathcal{Y}_{\mathcal{E}}B)} && \text{since } \tilde{F}FB \cong \mathcal{Y}_{\mathcal{E}}B \\ &\cong \widehat{\mathcal{E}_{/B}} && \text{since } \mathbb{G}(\mathcal{Y}_{\mathcal{E}}B) \cong \mathcal{E}_{/B}. \end{aligned}$$

Now calculating the effect of this equivalence explicitly, we see that it is simply the functor $((\tilde{F}F)_B)^{\sim}$ associated with the sliced functor $(\tilde{F}F)_B\colon \mathcal{E}_{/B} \to \hat{\mathcal{E}}_{/\tilde{F}FB}$. So it is easily checked that we have an (essentially) commutative triangle

$$\begin{tikzcd}
 & \hat{\mathcal{E}}_{/\tilde{F}FB} \arrow[d, "((\tilde{F}F)_B)^{\sim}", "\simeq"'] \\
\mathcal{X}_{/FB} \arrow[ur, "\tilde{F}_{FB}"] \arrow[dr, "\widetilde{F_B}"'] \arrow[r, phantom, "\cong"] & \widehat{\mathcal{E}_{/B}}
\end{tikzcd}$$

and it follows, from the fact that $\tilde{F}_{FB}$ is fully faithful, that $\widetilde{F_B}$ is fully faithful and thus that F_B is dense as required.

Having established these properties of the sliced versions of F let us return to our powerful arrow $p\colon E \to B$. Observe that for any pair of objects $f\colon D \to B$ in $\mathcal{E}_{/B}$ and $g\colon C \to E$ in $\mathcal{E}_{/E}$, we have a sequence of isomorphisms

$$\begin{aligned}
&\mathcal{X}_{/FB}(F_B(f), \Pi_{Fp}F_E(g)) && \\
&\quad\cong \mathcal{X}_{/FE}((Fp)^*F_B(f), F_E(g)) && Fp \text{ powerful so } (Fp)^* \dashv \Pi_{Fp} \\
&\quad\cong \mathcal{X}_{/FE}(F_E\, p^*(f), F_E(g)) && F \text{ preserves pullbacks so } (Fp)^*F_B \cong F_E\, p^* \\
&\quad\cong \mathcal{E}_{/E}(p^*(f), g) && F_E \text{ is fully faithful (see above)} \\
&\quad\cong \mathcal{E}_{/B}(f, \Pi_p(g)) && p \text{ powerful so } p^* \dashv \Pi_p \\
&\quad\cong \mathcal{X}_{/FB}(F_B(f), F_B\Pi_p(g)) && F_B \text{ is fully faithful.}
\end{aligned}$$

which are natural in f and g. In other words, by composing these we obtain isomorphisms $\widetilde{F_B}(F_B\Pi_p(g)) \cong \widetilde{F_B}(\Pi_{Fp}F_E(g))$ in $\widehat{\mathcal{E}_{/B}}$ which are themselves natural in the object $g \in \mathcal{E}_{/E}$. However F_B is dense, as demonstrated above, so $\widetilde{F_B}$ is fully faithful and it follows that there are unique isomorphisms $F_B\Pi_p(g) \cong \Pi_{Fp}F_E(g)$ which are mapped to the isomorphism of the last sentence by $\widetilde{F_B}$. These isomorphisms are again natural in $g \in \mathcal{E}_{/E}$ (since $\widetilde{F_B}$ is faithful) providing us with a natural isomorphism $F_B\Pi_p \cong \Pi_{Fp}F_E$. It is now a matter of routine computation to check that this latter isomorphism is indeed the mate described in Definition 2.7 and thus that F preserves p-products as postulated in the statement. ∎

2.11. Corollary. *The Yoneda embedding $よ_{\mathcal{E}}\colon \mathcal{E} \to \hat{\mathcal{E}}$ preserves internal p-products for all powerful morphisms p in $\mathcal{E}$.*

Proof. The Yoneda embedding is dense, fully faithful and left exact. Furthermore, all arrows are powerful in $\hat{\mathcal{E}}$. It follows that the conditions of the last proposition hold for the Yoneda embedding and any powerful arrow p in $\mathcal{E}$. So applying that proposition, we get the result of the statement. ∎

We are interested in finding some powerful morphisms in a functor category $[\mathcal{C},\mathcal{E}]$ in terms of those in $\mathcal{E}$. We are grateful to Steve Lack for vastly simplifying our proof of:

2.12. THEOREM. *Suppose $\mathcal{C}$ is a category and suppose $\mathcal{E}$ is finitely complete and has products indexed by the set $\mathcal{C}_1$ of morphisms of $\mathcal{C}$. A morphism $p\colon E \to B$ in the functor category $[\mathcal{C},\mathcal{E}]$ is powerful if it is componentwise powerful; that is, for all objects $U \in \mathcal{C}$, the component $pU\colon EU \to BU$ is powerful in $\mathcal{E}$.*

PROOF. Let $J\colon \mathcal{C}_0 \to \mathcal{C}$ denote the inclusion of the set of objects $\mathcal{C}_0$ as a discrete category. The hypothesis of the first sentence is used to obtain the functor

$$\begin{aligned} \mathrm{Ran}_J\colon\ [\mathcal{C}_0,\mathcal{E}] &\longrightarrow [\mathcal{C},\mathcal{E}]\,, \\ \mathrm{Ran}_J(T)U &= \prod_{V\in\mathcal{C}_0} (TV)^{\mathcal{C}(U,V)}, \end{aligned} \tag{7}$$

which is right Kan extension along J. Consider the following square of functors.

$$\begin{tikzcd} {[\mathcal{C},\mathcal{E}]_{/B}} \arrow[r, "p^*"] \arrow[d, "{[J,1]_B}"'] \arrow[dr, phantom, "\cong"] & {[\mathcal{C},\mathcal{E}]_{/E}} \arrow[d, "{[J,1]_E}"] \\ {[\mathcal{C}_0,\mathcal{E}]_{/BJ}} \arrow[r, "(pJ)^*"'] & {[\mathcal{C}_0,\mathcal{E}]_{/EJ}} \end{tikzcd} \tag{8}$$

The vertical functors are comonadic with right adjoints easily obtained from Ran_J and pullback along the unit. In particular, the units of the adjunctions are pointwise regular monomorphisms. The bottom functor has a right adjoint precisely because each component $pU\colon EU \to BU$ is powerful in $\mathcal{E}$. So Lemma 2.1 applies to yield a right adjoint for the top functor. So $p\colon E \to B$ is powerful. ■

2.13. THEOREM. *Let $\mathcal{C}$ be a category and let $\mathcal{E}$ and $\mathcal{F}$ be finitely complete categories which possess all products indexed by $\mathcal{C}_1$. Furthermore let $F\colon \mathcal{E} \to \mathcal{F}$ be a finite limit preserving functor which also preserves products indexed by $\mathcal{C}_1$. Finally let $p\colon E \to B$ be an arrow of the functor category $[\mathcal{C},\mathcal{E}]$.*

Now suppose that p and its composite Fp with F are both pointwise powerful. Furthermore assume that F preserves internal pU-products for each component $pU\colon EU \to BU$ of p. Then it follows that $[\mathcal{C},\mathcal{E}]$ has internal p-products, $[\mathcal{C},\mathcal{F}]$ has internal Fp-products and the functor $[1,F]\colon [\mathcal{C},\mathcal{E}] \to [\mathcal{C},\mathcal{F}]$ preserves internal p-products.

PROOF. Applying theorem 2.12 to the assumption that p and Fp are both pointwise powerful allows us to infer that these two maps are themselves powerful in $[\mathcal{C},\mathcal{E}]$ and $[\mathcal{C},\mathcal{F}]$ respectively. To show that $[1,F]$ preserves these internal p-products consider the following cube of functors and natural isomorphisms:

$$
\begin{tikzcd}
{[\mathcal{C},\mathcal{E}]_{/B}} \arrow[r, "p^*"] \arrow[d, "{[1,F]_B}"'] \arrow[dr, phantom, "\overset{(a)}{\cong}"] & {[\mathcal{C},\mathcal{E}]_{/E}} \arrow[d, "{[1,F]_E}"] \arrow[dr, "{[J,1]_E}"] & \\
{[\mathcal{C},\mathcal{F}]_{/FB}} \arrow[r, "(Fp)^*"] \arrow[dr, "{[J,1]_{FB}}"'] & {[\mathcal{C},\mathcal{F}]_{/FE}} \arrow[dr, "{[J,1]_{FE}}"] \arrow[d, phantom, "\overset{(c)}{\cong}"] \arrow[r, phantom, "\overset{(b)}{=}"] & {[\mathcal{C}_o,\mathcal{E}]_{/EJ}} \arrow[d, "{[1,F]_{EJ}}"] \\
& {[\mathcal{C}_0,\mathcal{F}]_{/FBJ}} \arrow[r, "(FpJ)^*"'] & {[\mathcal{C}_0,\mathcal{F}]_{/FEJ}}
\end{tikzcd}
=
\begin{tikzcd}
{[\mathcal{C},\mathcal{E}]_{/B}} \arrow[r, "p^*"] \arrow[d, "{[1,F]_B}"'] \arrow[dr, "{[J,1]_B}"] & {[\mathcal{C},\mathcal{E}]_{/E}} \arrow[dr, "{[J,1]_E}"] \arrow[d, phantom, "\overset{(d)}{\cong}"] & \\
{[\mathcal{C},\mathcal{F}]_{/FB}} \arrow[dr, "{[1,J]_{FB}}"'] \arrow[r, phantom, "\overset{(e)}{=}"] & {[\mathcal{C}_0,\mathcal{E}]_{/BJ}} \arrow[r, "(pJ)^*"'] \arrow[d, "{[1,F]_{BJ}}"] \arrow[dr, phantom, "\overset{(f)}{\cong}"] & {[\mathcal{C}_0,\mathcal{E}]_{/EJ}} \arrow[d, "{[1,F]_{EJ}}"] \\
& {[\mathcal{C}_0,\mathcal{F}]_{/FBJ}} \arrow[r, "(FpJ)^*"'] & {[\mathcal{C}_0,\mathcal{F}]_{/FEJ}}
\end{tikzcd}
\tag{9}
$$

Notice here that the commutativity of the squares marked (b) and (e) above goes without comment. To construct the remaining isomorphisms here, simply observe that the each of these squares contains a canonical natural transformation induced by the universal property of one of the horizontal pullback functors. The uniqueness of such induced maps ensures that the composites of the two faces of our cube are equal. Furthermore the pullbacks used to construct our pullback functors are all constructed pointwise and are thus preserved by the pre- and post-composition functors $[J,1]$ and $[1,F]$. This then is enough to show that each induced 2-cell is actually a natural isomorphism as drawn.

Now observe, as above, that by assumption each of the maps p, pJ, Fp and FpJ is pointwise powerful, so theorem 2.12 tells us that each one is powerful and thus that we have right adjoints (internal product) to each of the horizontal functors in display (9). Now, arguing as in the proof of that theorem we know that we may also use the finite limits of $\mathcal{E}$ or $\mathcal{F}$ to construct right adjoints (Kan extension) Ran_J to each of the pre-composition functors $[J,1]$, so it follows that each of the diagonal functors in display (9) has a right adjoint. Furthermore, just as in that proof we also know that the components of the units of these latter adjunctions are all regular monomorphisms.

Taking mates of the squares labelled (a), (b), (e) and (f) under the right adjoints to the various functors surrounding the squares labelled (c) and (d), and applying the usual properties of mates, we obtain the following commutative cube of natural transformations:

$$
\begin{tikzcd}
{[\mathcal{C},\mathcal{E}]_{/B}} \arrow[d, "{[1,F]_B}"'] \arrow[dr, phantom, "\Downarrow(a')"] & {[\mathcal{C},\mathcal{E}]_{/E}} \arrow[l, "\Pi_p"'] \arrow[d, "{[1,F]_E}"] & \\
{[\mathcal{C},\mathcal{F}]_{/FB}} & {[\mathcal{C},\mathcal{F}]_{/FE}} \arrow[l, "\Pi_{Fp}"'] \arrow[d, phantom, "\overset{(c')}{\cong}"] \arrow[r, phantom, "\Downarrow(b')"] & {[\mathcal{C}_o,\mathcal{E}]_{/EJ}} \arrow[ul, "(\mathrm{Ran}_J)_E"'] \arrow[d, "{[1,F]_{EJ}}"] \\
& {[\mathcal{C}_0,\mathcal{F}]_{/FBJ}} \arrow[ul, "(\mathrm{Ran}_J)_{FB}"] \arrow[r, "\Pi_{FpJ}"'] & {[\mathcal{C}_0,\mathcal{F}]_{/FEJ}} \arrow[ul, "(\mathrm{Ran}_J)_{FE}"]
\end{tikzcd}
=
\begin{tikzcd}
{[\mathcal{C},\mathcal{E}]_{/B}} \arrow[d, "{[1,F]_B}"'] \arrow[dr, phantom, "\Downarrow(e')"] & {[\mathcal{C},\mathcal{E}]_{/E}} \arrow[l, "\Pi_p"'] \arrow[d, phantom, "\overset{(d')}{\cong}"] & \\
{[\mathcal{C},\mathcal{F}]_{/FB}} & {[\mathcal{C}_0,\mathcal{E}]_{/BJ}} \arrow[ul, "(\mathrm{Ran}_J)_B"'] \arrow[d, "{[1,F]_{BJ}}"] \arrow[dr, phantom, "\Downarrow(f')"] & {[\mathcal{C}_0,\mathcal{E}]_{/EJ}} \arrow[l, "\Pi_{pJ}"] \arrow[ul, "(\mathrm{Ran}_J)_E"'] \arrow[d, "{[1,F]_{EJ}}"] \\
& {[\mathcal{C}_0,\mathcal{F}]_{/FBJ}} \arrow[ul, "(\mathrm{Ran}_J)_{FB}"] & {[\mathcal{C}_0,\mathcal{F}]_{/FEJ}} \arrow[l, "\Pi_{FpJ}"]
\end{tikzcd}
\tag{10}
$$

Now, just as in the proof of theorem 2.12 the internal pJ- and FpJ-products may be constructed pointwise, since $\mathcal{C}_0$ is discrete, so by applying the assumption that F preserves internal pU-products for each component pU of p we find that the 2-cell labelled (f') above is actually a natural isomorphism.

Turning now to squares (b') and (e'), observe that the Kan extension functors Ran_J are constructed using limits indexed by $\mathcal{C}_1$ in $\mathcal{E}$ and $\mathcal{F}$ and these are preserved by F, so it follows that $[1, F]\,\mathrm{Ran}_J \cong \mathrm{Ran}_J[1, F]$. Furthermore the functors $(\mathrm{Ran}_J)_B$ and $(\mathrm{Ran}_J)_E$ are constructed by applying Ran_J and taking a pullback along the component of the unit of $[J, 1] \dashv \mathrm{Ran}_J$ at B and E respectively. These pullbacks are constructed pointwise and so are also preserved by $[1, F]$. Finally we can combine these facts to demonstrate that the 2-cells in (b') and (e') are also natural isomorphisms, since their components are induced by a universal property which is held in common by their domains and codomains.

At this point, by composing the diagram of natural isomorphisms on the right of display (10) and then "cancelling" by the isomorphisms in squares (b') and (c') on its left hand side we find that the 2-cell obtained by pre-composing the cell in (a') by the functor $(\mathrm{Ran}_J)_E$ is itself a natural isomorphism. But notice that Π_p and Π_{Fp} both preserve finite limits, since they are right adjoints, as do $[1, F]_B$ and $[1, F]_E$ since limits are constructed pointwise in $[\mathcal{C}, \mathcal{E}]$ and the finite limits of $\mathcal{E}$ are preserved by F. Furthermore, we know that the unit of the adjunction $[J, 1] \dashv \mathrm{Ran}_J$ has components which are all regular monomorphisms, so we may apply the cancellation result of remark 2.2 to demonstrate that the 2-cell in (a') is itself a natural isomorphism. Finally, this simply tells us that $[1, F]$ preserves internal p-products as postulated. ■

Our next result concerns powerful morphisms in pseudopullbacks. Consider the following square (commutative up to isomorphism) made up at the vertices of categories admitting pullbacks and at the edges of pullback-preserving functors.

$$\begin{tikzcd} \mathcal{P} \arrow[r, "Q"] \arrow[d, "P"'] \arrow[dr, phantom, "\cong"] & \mathcal{B} \arrow[d, "G"] \\ \mathcal{A} \arrow[r, "F"'] & \mathcal{X} \end{tikzcd} \tag{11}$$

2.14. Proposition. *Assume that the diagram (11) is the pseudopullback of a pair of pullback-preserving functors F and G. Suppose the morphism $p\colon E \twoheadrightarrow B$ in $\mathcal{P}$ is such that:*

- *both Pp and Qp are powerful; and,*
- *the functors F and G preserve internal Pp-products and internal Qp-products, respectively.*

Then $p\colon E \twoheadrightarrow B$ is powerful in $\mathcal{P}$.

PROOF. The adjunction $p^* \dashv \Pi_p \colon \mathcal{P}_{/E} \to \mathcal{P}_{/B}$ is induced on pseudopullbacks by the adjunction

$$\begin{tikzcd}
\mathcal{A}_{/PE} \arrow[r, "F_{PE}"] \arrow[d, shift right=2, "\Pi_{Pp}"] & \mathcal{X}_{/FPE} \arrow[r, leftrightarrow, "\cong"] \arrow[d, shift right=2, "\Pi_{FPp}"] & \mathcal{X}_{/GQE} \arrow[d, shift right=2, "\Pi_{GQp}"] & \mathcal{B}_{/QE} \arrow[l, "G_{QE}"'] \arrow[d, shift right=2, "\Pi_{Qp}"] \\
\mathcal{A}_{/PB} \arrow[r, "F_{PB}"'] \arrow[u, shift right=2, "(Pp)^*"] \arrow[u, phantom, "\dashv"] \arrow[ur, phantom, "\cong"] & \mathcal{X}_{/FPB} \arrow[r, leftrightarrow, "\cong"'] \arrow[u, shift right=2, "(FPp)^*"] \arrow[u, phantom, "\dashv"] \arrow[ur, phantom, "\cong"] & \mathcal{X}_{/GQB} \arrow[u, shift right=2, "(GQp)^*"] \arrow[u, phantom, "\dashv"] \arrow[ur, phantom, "\cong"] & \mathcal{B}_{/QB} \arrow[l, "G_{QB}"] \arrow[u, shift right=2, "(Qp)^*"] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

between cospan diagrams. ∎

2.15. DEFINITION. *A morphism $p \colon E \to B$ in a 2-category $\mathcal{K}$ is called **representably powerful** when, for all objects X of $\mathcal{K}$, the functor $\mathcal{K}(X,p) \colon \mathcal{K}(X,E) \to \mathcal{K}(X,B)$ is powerful (see Example 2.8).*

Both fibrations and opfibrations in $\mathcal{K}$ (in the sense of [14]) are representably powerful since they are defined representably, and fibrations and opfibrations are powerful in Cat.

2.16. THEOREM. *A morphism $p \colon E \to B$ in $\mathrm{Cat}(\mathcal{E})$ is powerful if*

(a) the morphisms $p_i \colon E_i \to B_i$ in $\mathcal{E}$ are powerful for $i = 0, 1, 2$, and

(b) the morphism $p \colon E \to B$ in $\mathrm{Cat}(\mathcal{E})$ is representably powerful.

PROOF. Let $\Delta_{\restriction 2}$ denote the (finite) full subcategory of the usual category $\mathbf{\Delta}$ of non-empty finite ordinals consisting of the ordinals $[0], [1], [2]$ of cardinality ≤ 3. We have the following pseudopullback.

$$\begin{tikzcd}
\mathrm{Cat}(\mathcal{E}) \arrow[r, "\text{よ}"] \arrow[d] \arrow[dr, phantom, "\cong"] & {[\mathcal{E}^{\mathrm{op}}, \mathrm{Cat}]} \arrow[d, "{[\mathcal{E}^{\mathrm{op}}, N]}"] \\
{[\Delta_{\restriction 2}^{\mathrm{op}}, \mathcal{E}]} \arrow[r, "{[\Delta_{\restriction 2}^{\mathrm{op}}, \text{よ}_{\mathcal{E}}]}"'] & {[(\Delta_{\restriction 2} \times \mathcal{E})^{\mathrm{op}}, \mathrm{Set}]}
\end{tikzcd} \tag{12}$$

By Proposition 2.10, Theorem 2.13 and Theorem 2.12, the morphism $p \colon E \to B$ satisfies the conditions of Proposition 2.14. ∎

2.17. COROLLARY. *If $p \colon E \to B$ is either a fibration or opfibration in $\mathrm{Cat}(\mathcal{E})$ and p_0, p_1, p_2 are powerful in $\mathcal{E}$ then p is powerful in $\mathrm{Cat}(\mathcal{E})$.* ∎

2.18. COROLLARY. *A discrete (op)fibration $p \colon E \to B$ in $\mathrm{Cat}(\mathcal{E})$ is powerful if and only if $p_0 \colon E_0 \to B_0$ is powerful in $\mathcal{E}$.*

PROOF. In this case, p_1 and p_2 are pullbacks of p_0. The result follows from Corollary 2.6(3) and the fact that a right adjoint for $- \underset{B}{\times} E$ restricts to one for $- \underset{B_0}{\times} E_0$. ∎

Recall (from Section 6 of [15] for example) that an internal full subcategory of $\mathcal{E}$ is a category B in $\mathcal{E}$ together with a discrete opfibration $q\colon J \twoheadrightarrow B$ such that, for all objects X of $\mathcal{E}$, the functor

$$J_X\colon \mathcal{E}(X,B) \longrightarrow \mathcal{E}_{/X}\,, \qquad (13)$$
$$J_X(u) = \left(\mathrm{pr}_1\colon X \underset{B}{\times} J \twoheadrightarrow X\right)$$

is fully faithful. We say "B consists of the fibres of $q_0\colon J_0 \twoheadrightarrow B_0$".

2.19. Corollary. *If $f\colon M \twoheadrightarrow K$ is a powerful morphism in $\mathcal{E}$ then the internal full subcategory $q\colon J \twoheadrightarrow B$ of $\mathcal{E}$ consisting of the fibres of f exists and is a powerful morphism in* $\mathrm{Cat}(\mathcal{E})$.

Proof. By Corollary 2.6(3), the morphism $f \times 1\colon M \times K \twoheadrightarrow K \times K$ is powerful in $\mathcal{E}$. By Proposition 2.4(3), the cartesian internal hom $(s,t)\colon B_1 \twoheadrightarrow K \times K$ of the two objects $f \times 1\colon M \times K \twoheadrightarrow K \times K$ and $1 \times f\colon K \times M \twoheadrightarrow K \times K$ exists in the slice category $\mathcal{E}_{/K\times K}$. It is easy to see that this property of $B_1 \twoheadrightarrow K \times K$ is equivalent to a natural isomorphism

$$\mathcal{E}_{/K\times K}(X \xrightarrow{(u,v)} K \times K, B_1 \xrightarrow{(s,t)} K \times K) \cong \mathcal{E}_{/X}(u^*(f), v^*(f)). \qquad (14)$$

The underlying graph of the category B is

$$B_1 \underset{t}{\overset{s}{\rightrightarrows}} K\,.$$

From (14) we see that the graph $\mathcal{E}(X,B)$ is isomorphic to the underlying graph of the full subcategory of $\mathcal{E}_{/X}$ consisting of objects of the form $u^*(f)$. So B is a full subcategory of $\mathcal{E}$ and we have our fully faithful pseudonatural transformation $\mathcal{E}(-,B) \twoheadrightarrow \mathcal{E}_{/-}$. Recall that the generalized Yoneda lemma of [15] gives an explicit equivalence

$$\mathrm{Hom}(\mathcal{E}^{\mathrm{op}}, \mathrm{Cat})\left(\mathcal{E}(-,B), \mathcal{E}_{/-}\right) \simeq \mathcal{E}^B \qquad (15)$$

where the right-hand side is the full subcategory of $\mathrm{Cat}(\mathcal{E})_{/B}$ consisting of the discrete opfibrations into B. Using this, we obtain a discrete opfibration $q\colon J \twoheadrightarrow B$ inducing $\mathcal{E}(-,B) \twoheadrightarrow \mathcal{E}_{/-}$ as J_- as in (13) and with $q_0 = f$ which is powerful. ∎

3. Internal comprehensive factorization

Let $\mathcal{E}$ denote a finitely complete and finitely cocomplete category. Let $\mathcal{K}$ denote the 2-category $\mathrm{Cat}(\mathcal{E})$ of categories in $\mathcal{E}$. We identify $\mathcal{E}$ with the sub-2-category of $\mathcal{K}$ consisting of the discrete categories in $\mathcal{E}$.

For each object A of $\mathcal{K}$, the coequalizer

$$A_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} A_0 \longrightarrow \pi_0(A) \qquad (16)$$

in $\mathcal{E}$ yields a coidentifier (in the sense of [16])

$$\begin{tikzcd} A^{\mathbf{2}} \arrow[r, bend left, "d_0"] \arrow[r, bend right, "d_1"'] \arrow[r, phantom, "\Downarrow\lambda"] & A \arrow[r] & \pi_0(A) \end{tikzcd} \tag{17}$$

in $\mathcal{K}$. Then $\pi_0\colon \mathcal{K} \to \mathcal{E}$ is left adjoint to the inclusion 2-functor. This implies that, if there is a 2-cell $f \Rightarrow g$ in $\mathcal{K}$, then $\pi_0(f) = \pi_0(g)$ in $\mathcal{E}$; so that adjoint morphisms in $\mathcal{K}$ are taken by π_0 to isomorphisms.

Using Day's Reflection Theorem (see Theorems (3.10) and (3.24) of [15]), we deduce the following.

3.1. PROPOSITION. *If $\mathcal{E}$ is cartesian closed then π_0 preserves finite products.* ■

3.2. PROPOSITION. *If $p\colon E \to B$ is a discrete fibration and $b\colon X \to B$ has discrete domain X in $\mathcal{K}$ then the composite*

$$Eb = E \underset{B}{\times} X \to (b/p) \to \pi_0(b/p) \tag{18}$$

is an isomorphism.

PROOF. Since p is a fibration, the inclusion $Eb \to (b/p)$ of the pullback into the comma object has a right adjoint. Since both X and p are discrete, $\pi_0(Eb) = Eb$. ■

3.3. DEFINITION. *A morphism $j\colon A \to B$ in $\mathcal{K}$ is said to be* ***final*** *when, for all powerful discrete opfibrations $q\colon F \to B$, the projection $A \underset{B}{\times} F \to F$ is inverted by the 2-functor π_0.*

3.4. PROPOSITION. *Suppose $j\colon A \to B$ is final. A morphism $k\colon B \to C$ is final if and only if kj is final.*

PROOF. If $q\colon F \to C$ is a powerful discrete opfibration, so too is $B \underset{C}{\times} F \to B$ (using Corollary 2.6(3)). Since j is final, it follows that π_0 inverts $A \underset{C}{\times} F \to B \underset{C}{\times} F$. So π_0 inverts $A \underset{C}{\times} F \to B \underset{C}{\times} F \to F$ if and only if it inverts $B \underset{C}{\times} F \to F$. ■

3.5. PROPOSITION. *Right adjoints are final.*

PROOF. Suppose $j\colon A \to B$ has a left adjoint and $q\colon F \to B$ is a powerful discrete opfibration. Let $g\colon P \to F$ be the pullback of j along q. Since j has a left adjoint, so too does the projection $(q/j) \to F$. Yet g factors through $(q/j) \to F$ via a morphism $P \to (q/j)$ which has a left adjoint, since q is an opfibration. So g has a left adjoint and hence induces $\pi_0(P) \cong \pi_0(F)$. ■

3.6. PROPOSITION. *Coidentifiers are final.*

PROOF. Suppose $j\colon A \to B$ is a coidentifier of some 2-cell and $q\colon F \to B$ is a powerful discrete opfibration. Then pullback with q has a right adjoint and so preserves coidentifiers. So $A \underset{B}{\times} F \to F$ is a coidentifier in $\mathcal{K}$. Since $\pi_0\colon \mathcal{K} \to \mathcal{E}$ is a left adjoint, it preserves coidentifiers. Yet coidentifiers in $\mathcal{E}$ are invertible! So π_0 inverts $A \underset{B}{\times} F \to F$. ■

3.7. Definition. *An object B of $\mathcal{K}$ is called* ***amenable*** *when $d_1\colon B_1 \to B_0$ is powerful in $\mathcal{E}$.*

3.8. Proposition. *If $p\colon E \to B$ is a discrete fibration and B is amenable then E is amenable.*

Proof. $d_1\colon E_1 \to E_0$ is a pullback of $d_1\colon B_1 \to B_0$ along p_0. So the result follows from Corollary 2.6(3). ∎

3.9. Proposition. *Each final discrete fibration into an amenable object is invertible.*

Proof. Suppose $p\colon E \to B$ is a final discrete fibration with B amenable. Furthermore let $i\colon B_0 \to B$ be the inclusion. Then $d_1\colon (i/B) \to B$ is a powerful discrete opfibration by Corollary 2.18. So the pullback $g\colon (i/p) \to (i/B)$ of p along $d_1\colon (i/B) \to B$ is inverted by π_0. Using Proposition 3.2, we see that $\pi_0(g)$ is isomorphic to $p_0\colon E_0 \to B_0$. A discrete fibration p with p_0 invertible is clearly invertible. ∎

3.10. Proposition. *If B is amenable then the category $\mathcal{E}^{B^{\mathrm{op}}}$ of discrete fibrations over B is finitely cocomplete.*

Proof. $\mathcal{E}^{B^{\mathrm{op}}}$ is monadic over $\mathcal{E}_{/B_0}$, and the functor underlying the monad is the composite

$$\mathcal{E}_{/B_0} \xrightarrow{{d_1}^*} \mathcal{E}_{/B_1} \xrightarrow{\Sigma_{d_0}} \mathcal{E}_{/B_0}\,. \tag{19}$$

If B is amenable this functor has a right adjoint and so preserves colimits. So $\mathcal{E}^{B^{\mathrm{op}}}$ has whatever colimits $\mathcal{E}$ has. ∎

Let $\mathcal{K}^{B^{\mathrm{op}}}$ denote the sub-2-category of $\mathcal{K}_{/B}$ consisting of the split fibrations into B and split-cartesian morphisms between them. The inclusion 2-functor $\mathcal{K}^{B^{\mathrm{op}}} \to \mathcal{K}_{/B}$ is monadic and its left adjoint takes $f\colon A \to B$ to $d_0\colon (B/f) \to B$. Recall (see Theorem 4.13 of [15]) that there is an equivalence of 2-categories

$$\mathcal{K}^{B^{\mathrm{op}}} \simeq \mathrm{Cat}(\mathcal{E}^{B^{\mathrm{op}}}). \tag{20}$$

We identify $\mathcal{E}^{B^{\mathrm{op}}}$ with the sub-2-category of $\mathcal{K}^{B^{\mathrm{op}}}$ consisting of the discrete objects.

If B is amenable, it follows from Proposition 3.10 that we can apply the construction of (17) to $\mathcal{E}^{B^{\mathrm{op}}}$ and $\mathcal{K}^{B^{\mathrm{op}}}$ in place of $\mathcal{E}$ and $\mathcal{K}$ to obtain the left adjoint

$$\pi_{0B}\colon \mathcal{K}^{B^{\mathrm{op}}} \to \mathcal{E}^{B^{\mathrm{op}}} \tag{21}$$

to the inclusion 2-functor defined by the coidentifier

$$\begin{tikzcd} (E,p)^{\mathbf{2}} \arrow[r, bend left, "d_0"] \arrow[r, bend right, "d_1"'] \arrow[r, phantom, "\Downarrow\lambda"] & E \arrow[r] & \pi_{0B}(E) \end{tikzcd} \tag{22}$$

for each split fibration $p\colon E \to B$, where $(E,p)^{\mathbf{2}}$ denotes the cotensor in $\mathcal{K}_{/B}$ constructed by pulling back $p^{\mathbf{2}}\colon E^{\mathbf{2}} \to B^{\mathbf{2}}$ along the diagonal $B \to B^{\mathbf{2}}$.

The composite 2-functor

$$\mathcal{E}^{B^{\mathrm{op}}} \to \mathcal{K}^{B^{\mathrm{op}}} \to \mathcal{K}_{/B} \tag{23}$$

is fully faithful (see subsection (2.9) of [15]). If B is amenable, it follows from (20) and (21) the composite 2-functor (23) has a left adjoint whose counit at $f\colon A \to B$ is the composite

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] \arrow[dr, phantom, "="] & (B/f) \arrow[r, "n"] \arrow[d, "d_0"'] \arrow[dr, phantom, "="] & \pi_{0B}(B/f) \arrow[d, "p"] \\ B \arrow[r, "1"'] & B \arrow[r, "1"'] & B \end{tikzcd} \tag{24}$$

where i has a left adjoint $d_1\colon (B/f) \to A$ and n is the coidentifier of the 2-cell

$$\begin{tikzcd} (d_1/A) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\lambda"] & (B/f) \end{tikzcd} \tag{25}$$

3.11. THEOREM. *Suppose $\mathcal{E}$ is a finitely complete and finitely cocomplete category. For each amenable object B of $\mathcal{K} = \mathrm{Cat}(\mathcal{E})$:*

(1) each morphism $f\colon A \to B$ in $\mathcal{K}$ factors as $f = pj$ where p is a discrete fibration and j is final;

(2) for all discrete fibrations $p\colon E \to C$ and final $j\colon A \to B$, the following square of functors is a pullback.

$$\begin{tikzcd} \mathcal{K}(B,E) \arrow[r, "\mathcal{K}(1,p)"] \arrow[d, "\mathcal{K}(j,1)"'] \arrow[dr, phantom, "="] & \mathcal{K}(B,C) \arrow[d, "\mathcal{K}(j,1)"] \\ \mathcal{K}(A,E) \arrow[r, "\mathcal{K}(1,p)"'] & \mathcal{K}(A,C) \end{tikzcd} \tag{26}$$

PROOF.

(1) In (24) $f = pni$ where p is a discrete fibration. Also, i has a left adjoint and n is a coidentifier. By Propositions 3.4, 3.5 and 3.6, it follows that $j = ni$ is final.

(2) Since $\mathcal{K}$ admits $(-)^{\mathbf{2}}$, the pullback condition only needs to be verified on objects. By Propositions 3.4 and 3.9, the reflection of a final $j\colon A \to B$ into the discrete fibrations over B is invertible. So $j\colon j \to 1_B$ in $\mathcal{K}_{/B}$ has the property of the unit at j for the reflection. This means that, for all $u\colon j \to p$ with p a discrete fibration into B, there exists a unique $w\colon 1_B \to p$ with $wj = u$.

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "u"'] & B \arrow[d, "1"] \arrow[dl, "w"] \\ E \arrow[r, "p"'] & B \end{tikzcd} \tag{27}$$

We must show more generally that, for all final $j\colon A \to B$ and discrete fibrations $p\colon E \to C$, if $pu = vj$ then there exists a unique w with $pw = v$ and $wj = u$. Form the pullback P of p and v. Let $u'\colon A \to P$ have projections u and j from the pullback. Note that $j = p'u'$ where $p' = \mathrm{pr}_2$ is a discrete fibration using Corollary 2.6(3). By (27), there exists a unique $w'\colon B \to P$ such that $p'w' = 1$ and $w'j = u'$. So there exists $w = \mathrm{pr}_1 w'$ satisfying $pw = v$ and $wj = u$. Finally, uniqueness is a consequence of the uniqueness of w' since any such w would induce a w' with $p'w' = 1$ and $w = \mathrm{pr}_1 w'$, which imply the condition $w'j = u'$. ∎

3.12. Corollary. *If B is amenable, the following conditions on $j\colon A \to B$ are equivalent:*

(a) the morphism j in $\mathcal{K}$ is final;

(b) the object $\pi_{0B}(B/j)$ of $\mathcal{K}_{/B}$ is terminal;

(c) the following diagram in $\mathcal{K}$ is a coidentifier

$$\begin{tikzcd} (d_1/A) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\lambda"] & (B/j) \arrow[r, "d_0"] & B \end{tikzcd}$$

(d) any pullback of j along a powerful opfibration in $\mathcal{K}$ is inverted by π_0. ∎

3.13. Corollary. *Any pullback along a powerful opfibration of a final morphism into an amenable object in $\mathcal{K}$ is final.* ∎

4. Torsors

Let $\mathcal{E}$ be a finitely complete category and let $\mathcal{K} = \mathrm{Cat}(\mathcal{E})$ be the 2-category of categories in $\mathcal{E}$.

4.1. Definition. *A discrete fibration $p\colon E \to B$ in $\mathcal{K}$ is said to be* ***representable*** *when there exist a morphism $x\colon 1 \to B$ and an isomorphism $(B/x) \cong E$ over B.*

Suppose $p\colon E \to B$ is a discrete fibration in $\mathcal{K}$ and R is an object of $\mathcal{E}$. Then $p \times R\colon E \times R \to B \times R$ is a discrete fibration in the 2-category $\mathcal{K}_{/R}$ of categories in $\mathcal{E}_{/R}$. When $p \times R\colon E \times R \to B \times R$ is representable, one says p is ***representable over*** *R.*

Suppose B is a category in $\mathcal{E}$. A ***B-torsor*** *is a discrete fibration $p\colon E \to B$ in $\mathcal{K}$ for which there exists an object R of $\mathcal{E}$ with $R \to 1$ a regular epimorphism and p representable over R.*

4.2. Proposition. *The following conditions on a discrete fibration $p\colon E \to B$ and an object R of $\mathcal{E}$ are equivalent:*

(i) the discrete fibration p is representable over R;

(ii) there exists a morphism $y\colon R \to E$ and a 2-cell $\chi\colon \mathrm{pr}_1 \Rightarrow y\,\mathrm{pr}_2\colon E\times R \to E$ whose restriction along $(y,1)\colon R \to E\times R$ is the identity 2-cell of y;

(iii) the category $\mathcal{K}(R,E)$ has a terminal object which is preserved by composition with all $u\colon R' \to R$ in $\mathcal{E}$;

(iv) (for R powerful in $\mathcal{E}$) the morphism $E^R \to 1$ in $\mathcal{K}$ has a right adjoint.

Furthermore, the pair (y,χ) is unique up to isomorphism when it exists.

Proof. To say that p is representable over R is to say that there exists $x\colon R \to B$ and a morphism of spans

$$t\colon\ (d_0,(B/x),d_1) \to (p\,\mathrm{pr}_1, E\times R, \mathrm{pr}_2) \tag{28}$$

from R to B with t invertible. By Yoneda's Lemma in a 2-category, morphisms of spans t are in bijection with morphisms of spans

$$k\colon\ (x,R,1) \to (p\,\mathrm{pr}_1, E\times R, \mathrm{pr}_2)\,. \tag{29}$$

Such a k must have the form $k=(y,1)$ for a unique y with $py=x$. On the other hand, morphisms of spans

$$t'\colon\ (p\,\mathrm{pr}_1, E\times R, R) \to (d_0,(B/x),d_1) \tag{30}$$

are in bijection with 2-cells $\mu\colon p\,\mathrm{pr}_1 \Rightarrow x\,\mathrm{pr}_2\colon E\times R \to B$ via the universal property of (B/x). Since p is a discrete fibration, there exists a unique 2-cell χ with $p\chi=\mu$. The condition $tt'=1$ becomes $\mathrm{pr}_1\,tt'=\mathrm{pr}_1$ and $\mathrm{pr}_2\,tt'=\mathrm{pr}_2$. The second of these is automatic while the first means that χ has domain pr_1. The condition $t't=1$ becomes $\mu(y,1)=1_x$ which amounts to $\chi(y,1)=1_y$ since p is discrete. This proves (i)⇔(ii).

Assuming (ii), we easily see that yu is terminal in $\mathcal{K}(R',E)$ which gives (iii).

It is easy to see that (iv) amounts to saying that there exists $y\colon R \to E$ such that, for all X in $\mathcal{E}$, the composite $y\,\mathrm{pr}_2\colon X\times R \to E$ is terminal in $\mathcal{K}(X\times R,E)$. So certainly (iii)⇒(iv). Yet (iv) easily implies that, for all A in $\mathcal{K}$, the composite $y\,\mathrm{pr}_2\colon A\times R \to E$ is terminal in $\mathcal{K}(A\times R,E)$. With $A=E$, there is therefore a unique $\chi\colon \mathrm{pr}_1 \Rightarrow y\,\mathrm{pr}_2\colon E\times R \to E$. With $X=1$, we see that y has only one endomorphism; thus $\chi(y,1)=1$. So finally (iv)⇒(ii). ∎

4.3. Remark. Point (ii) of the last proposition could equally well have been written as:

(ii)′ the morphism $\mathrm{pr}_2\colon (E\times R \xrightarrow{\mathrm{pr}_2} R) \to (R \xrightarrow{1} R)$ of the slice 2-category $\mathcal{K}_{/R}$ has a right adjoint in there.

In other words, this says that if we localise E over the cover $R \to 1$ then the resulting internal category over R possesses a terminal object.

The equivalence of (ii) and (ii)′ becomes clear on observing that morphisms $y\colon R \to E$ in $\mathcal{K}$, as in (ii), stand in bijective correspondance with morphisms $y'\colon (R \xrightarrow{\mathrm{id}_R} R) \to (E\times R \xrightarrow{\mathrm{pr}_2} R)$ in $\mathcal{K}_{/R}$, under a bijection which carries y to $(y,1)\colon R \to E\times R$. Furthermore

2-cells $\chi\colon \mathrm{pr}_1 \Rightarrow y\,\mathrm{pr}_2\colon E\times R \to E$ in $\mathcal{K}$, as in (ii), also stand in bijective correspondance with 2-cells $\chi'\colon 1 \Rightarrow y'\,\mathrm{pr}_2\colon (E\times R \xrightarrow{\mathrm{pr}_2} R) \to (E\times R \xrightarrow{\mathrm{pr}_2} R)$ in $\mathcal{K}_{/R}$, under a bijection which carries χ to $(\chi,1)$. Under these bijections, the condition on y and χ given in (ii) is equivalent to the statement that the triangle identities hold for an adjunction $\mathrm{pr}_2 \dashv (y,1)$ in $\mathcal{K}_{/R}$ whose unit is $(\chi,1)$ and whose counit is the identity on $R \xrightarrow{1} R$.

4.4. COROLLARY. *If one discrete fibration with domain E is representable over R then all are.* ∎

4.5. DEFINITION. *For each object R of $\mathcal{E}$, let R_c denote the* ***chaotic*** *(also called "coarse" or "indiscrete") category in $\mathcal{E}$ as displayed in the diagram:*

$$\begin{tikzcd} R\times R\times R \arrow[r, shift left=2, "\mathrm{pr}_1"] \arrow[r, "\mathrm{pr}_2"] \arrow[r, shift right=2, "\mathrm{pr}_3"'] & R\times R \arrow[r, shift left=2, "\mathrm{pr}_1"] \arrow[r, shift right=2, "\mathrm{pr}_2"'] & R \arrow[l, "\delta"'] \end{tikzcd} \tag{31}$$

A morphism $R_c \to B$ is termed a ***Čech cocycle****, defined over R, with coefficients in B.*

4.6. DEFINITION. *Let* $\mathrm{Tors}(R,B)$ *denote the full subcategory of $\mathcal{E}^{B^{\mathrm{op}}}$ consisting of the objects which are the discrete fibrations in $\mathcal{K}$ representable over R.*

There is a functor

$$\Theta = \Theta_R\colon \mathrm{Tors}(R,B) \to \mathcal{K}(R_c,B) \tag{32}$$

defined as follows. Take a discrete fibration $p\colon E \to B$ which is representable over R. Let y and χ be as in Proposition 4.2(ii). Let $j\colon R_c \to E$ denote the following morphism of graphs.

$$\begin{tikzcd} R\times R \arrow[r, "\chi(y\times 1)"] \arrow[d, shift right=2, "\mathrm{pr}_1"'] \arrow[d, shift left=2, "\mathrm{pr}_2"] \arrow[dr, phantom, "="] & E_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\ R \arrow[r, "y"'] & E_0 \end{tikzcd} \tag{33}$$

For all 2-cells $\xi\colon e \Rightarrow e'\colon X \to E$ and all $r\colon X \to R$, we have that the composite

$$e = \mathrm{pr}_1(e,r) \xrightarrow{\chi(e,r)} y\,\mathrm{pr}_2(e,r) = yr \xrightarrow{j\,\mathrm{pr}_2(\xi,1)=1} y\,\mathrm{pr}_2(e',r) = yr$$

is equal to the composite

$$e = \mathrm{pr}_1(e,r) \xrightarrow{\xi=\mathrm{pr}_2(\xi,1)} e' = \mathrm{pr}_1(e',r) \xrightarrow{\chi(e',r)} y\,\mathrm{pr}_2(e',r) = yr.$$

This gives $\chi(yr,t) = \chi(ys,t)\,\chi(yr,s)$ which, together with $\chi(y,1)=1$, means that j is a functor. It also implies that χ extends to a 2-cell

$$\chi\colon \mathrm{pr}_1 \Rightarrow j\,\mathrm{pr}_2\colon E\times R_c \to E. \tag{34}$$

Define

$$\Theta_R(p) = pj\colon R_c \to B. \tag{35}$$

For any morphism $w\colon E \to E'$ in $\mathrm{Tors}(R, B)$, construct j' and χ' from E' just as we constructed j and χ from E. Define

$$\Theta_R(w)\colon pj \Rightarrow p'j'\colon R_c \to B$$

to be the 2-cell

$$pj = p' \operatorname{pr}_1(wj, 1) \xrightarrow{p'\chi'(wj,1)} p'j' \operatorname{pr}_2(wj, 1) = p'j'.$$

4.7. PROPOSITION. *Suppose $\mathcal{E}$ is cartesian closed and $R \to 1$ is a regular epimorphism in $\mathcal{E}$. If $p\colon E \to B$ is a discrete fibration which is representable over R then the morphism $j\colon R_c \to E$ defined in (33) is final.*

PROOF. Take a discrete opfibration $q\colon F \to E$ (not necessarily powerful) and form the pullback

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "h"'] \arrow[dr, phantom, "="] & F \arrow[d, "q"] \\ R_c \arrow[r, "j"'] & E \end{tikzcd} .$$

Let $\phi\colon \operatorname{pr}_1 \Rightarrow w$ be the unique 2-cell with

$$q\phi = \chi\,(q \times 1_{R_c}) : q \operatorname{pr}_1 = \operatorname{pr}_1(q \times 1_{R_c}) \Rightarrow j \operatorname{pr}_2(q \times 1_{R_c}) = qw.$$

From the pullback property, there exists a unique $u\colon F \times R_c \to P$ with $hu = \operatorname{pr}_2$ and $gu = w$. Since $R \to 1$ is a regular epimorphism, we have the coequalizer

$$\begin{tikzcd} R \times R \arrow[r, shift left, "\operatorname{pr}_1"] \arrow[r, shift right, "\operatorname{pr}_2"'] & R \arrow[r] & 1 \end{tikzcd}$$

so $\pi_0(R_c) \cong 1$. By Proposition 3.1,

$$\pi_0(\operatorname{pr}_1)\colon \pi_0(F \times R_c) \cong \pi_0(F) \times \pi_0(R_c) \to \pi_0(F)$$

is invertible. Since we have $\phi\colon \operatorname{pr}_1 \Rightarrow w$, it follows that $\pi_0(w)$ is invertible. Notice that

$$q\phi\,(g, h) = \chi\,(q \times 1)\,(g, h) = \chi\,(qg, h) = \chi\,(jh, h) = 1;$$

so there exists a unique $\beta\colon 1_P \Rightarrow u(g, h)$ such that $h\beta = 1$ and $g\beta = \phi(g, h)$. Thus $1 = \pi_0(u)\pi_0(g, h)$ and $\pi_0(u)$ is a retraction. But $\pi_0(g)\pi_0(u) = \pi_0(w)$ is invertible. So $\pi_0(g)$ is invertible, as required. ∎

4.8. Proposition. *Suppose $\mathcal{E}$ is a cartesian closed and finitely cocomplete category. Suppose $p\colon E \to B$ is a discrete fibration with E amenable and suppose R is an object of $\mathcal{E}$. If there exists a final $j\colon R_c \to E$ then p is representable over R; moreover, j is unique up to isomorphism.*

Proof. The effect of j on objects gives a morphism $y\colon R \to E$ and the effect on morphisms gives a 2-cell

$$\phi\colon j\,\mathrm{pr}_1 \Rightarrow y\,\mathrm{pr}_2\colon R_c \times R \to E.$$

This last 2-cell induces a functor $h\colon R_c \times R \to (E/y)$ with $d_0 h = j\,\mathrm{pr}_1$, $d_1 h = \mathrm{pr}_2$, and $\lambda h = \phi$. Since $\mathcal{E}$ is cartesian closed, $R \to 1$ is powerful. So $\mathrm{pr}_1\colon E \times R \to E$ is a powerful discrete opfibration (using Corollary 2.6(3)). Consequently the top morphism in the following square is final.

$$\begin{tikzcd} R_c \times R \arrow[r, "j\times 1"] \arrow[d, "h"'] \arrow[dr, phantom, "="] & E \times R \arrow[d, "\mathrm{pr}_1"] \\ (E/y) \arrow[r, "d_0"'] & E \end{tikzcd}$$

Since E is amenable, so too is $E\times R$. By Theorem 3.11(2), there exists a unique morphism $x\colon E \times R \to (E/j)$ such that $x(j \times 1) = h$ and $d_0 x = \mathrm{pr}_1$. The two morphisms $d_1 x$ and $\mathrm{pr}_2\colon E \times R \to R$ determine a 2-cell

$$\theta\colon d_1 x \Rightarrow \mathrm{pr}_2\colon E \times R \to R_c.$$

Define χ to be the composite 2-cell

$$\chi\colon\ \mathrm{pr}_1 = d_0 x \xRightarrow{\lambda x} y d_1 x \xRightarrow{j\theta} y\,\mathrm{pr}_2\,.$$

Using $x(j\times 1) = h$, we deduce that $\chi(y, 1) = 1$. So p is representable over R by Proposition 4.2. Notice that j arises from y and x via the construction (33); from uniqueness of y and x (Proposition 4.2), we deduce that of j. ∎

4.9. Theorem. *Suppose $\mathcal{E}$ is finitely complete, finitely cocomplete and cartesian closed. Suppose $R \to 1$ is a regular epimorphism in $\mathcal{E}$ and B is an amenable category in $\mathcal{E}$. Then the functor*

$$\Theta_R\colon\ \mathrm{Tors}(R, B) \to \mathcal{K}(R_c, B)$$

of (32) is an equivalence of categories.

Proof. Take any $f\colon R_c \to B$ and factorize it as $f = pj$ as in Theorem 3.11(a). By Proposition 4.8, p is an object of $\mathrm{Tors}(R, B)$. Then $\Theta_R(p) = pj'$ for some final j' by Proposition 4.7. By uniqueness in 4.8, $j \cong j'$. So $\Theta_R(p) \cong pj = f$.

Suppose $\omega\colon \Theta_R(p) \Rightarrow \Theta_R(p')$ is a morphism in $\mathcal{K}(R_c, B)$. Let $\Theta_R(p) = pj$ and $\Theta_R(p') = p'j'$ where j and j' are final. Since p' is a discrete fibration, there exists a unique $\phi\colon u \Rightarrow j'$ with $p'\phi = \omega$. Since $pj = p'u$ and E is amenable (by Proposition 3.8), Theorem 3.11(b) gives a unique w with $wj = u$ and $p'w = p$. Thus we have $w\colon p \Rightarrow p'$ in $\mathrm{Tors}(R, B)$ which is unique with $\Theta_R(p) = \omega$. ∎

Let $\mathcal{R}$ denote the ordered set whose elements are the objects R of $\mathcal{E}$ for which $R \twoheadrightarrow 1$ is a regular epimorphism; the order is

$$R' \leq R \text{ if and only if there exists } R' \twoheadrightarrow R \text{ in } \mathcal{E}.$$

If $R' \leq R$ then $\mathrm{Tors}(R, B)$ is a full subcategory of $\mathrm{Tors}(R', B)$. So we have a functor

$$\mathrm{Tors}(-, B)\colon \mathcal{R}^{\mathrm{op}} \twoheadrightarrow \mathrm{Cat}\,. \tag{36}$$

4.10. Proposition. *Assume $\mathcal{E}$ is cartesian closed. Then $\mathcal{R}^{\mathrm{op}}$ is a directed ordered set. Moreover, if $u\colon R' \twoheadrightarrow R$ is a morphism with R and R' in $\mathcal{R}$, then $u_c\colon R'_c \twoheadrightarrow R_c$ is final.*

Proof. Since $X \times -$ preserves coequalizers, if R and R' are in $\mathcal{R}$, so is $R \times R'$. This proves the first sentence. There is a unique 2-cell $\chi\colon \mathrm{pr}_1 \Rightarrow u\,\mathrm{pr}_2\colon R_c \times R' \twoheadrightarrow R_c$ which (by Proposition 4.2) implies $1_{R_c}\colon R_c \twoheadrightarrow R_c$ is representable over R'. So $\Theta_{R'}(1_{R_c}) = u_c$ is final by Proposition 4.7. ∎

Let $\mathrm{Tors}(B)$ denote the full subcategory of $\mathcal{E}^{B^{\mathrm{op}}}$ consisting of those discrete fibrations which are representable over some R in $\mathcal{R}$. From Proposition 4.10 we have a filtered colimit

$$\mathrm{Tors}(B) \cong \operatorname*{colim}_{R\in\mathcal{R}^{\mathrm{op}}} \mathrm{Tors}(R, B). \tag{37}$$

Moreover, we obtain a pseudofunctor (homomorphism of bicategories)

$$\mathcal{K}(-_c, B)\colon \mathcal{R}^{\mathrm{op}} \twoheadrightarrow \mathrm{Cat} \tag{38}$$

whose value at R is $\mathcal{K}(R_c, B)$ and whose value at $R' \leq R$ is composition with some $u_c\colon R'_c \twoheadrightarrow R_c$. Then the functors Θ_R of (32) become the components of a pseudonatural transformation in $R \in \mathcal{R}^{\mathrm{op}}$. From Theorem 4.9 and the isomorphism (37), we deduce the following result on the construction of torsors from Čech cocycles.

4.11. Theorem. [Clutching] *If $\mathcal{E}$ is finitely complete, finitely cocomplete, and cartesian closed, and B is amenable, the functors (32) induce an equivalence of categories:*

$$\mathrm{Tors}(B) \simeq \operatorname*{bicolim}_{R\in\mathcal{R}^{\mathrm{op}}} \mathcal{K}(R_c, B)$$

∎

5. Classification of objects locally isomorphic to some member of a given internal family

Let $\mathcal{E}$ denote a finitely complete category and let $\mathcal{F} = \mathrm{Hom}(\mathcal{E}^{\mathrm{op}}, \mathrm{Cat})$ denote the 2-category of homomorphisms of bicategories (pseudofunctors) from $\mathcal{E}^{\mathrm{op}}$ to Cat, strong transformations, and modifications. Recall from [17] that $\mathcal{F}$ is a cartesian closed bicategory in the sense that it has an internal hom Y^X which appears in an equivalence of categories

$$\mathcal{F}(Z, Y^X) \simeq \mathcal{F}(X \times Z, Y).$$

Each category B in $\mathcal{E}$ will be identified with $\mathcal{E}(-, B)\colon \mathcal{E}^{\mathrm{op}} \twoheadrightarrow \mathrm{Cat}$ in $\mathcal{F}$.

5.1. Definition. *We identify $\mathcal{K} = \mathrm{Cat}(\mathcal{E})$ with its image in $\mathcal{F}$ under this embedding $\mathcal{K} \to \mathcal{F}$. Objects in this image are, of course, called* ***categories internal to $\mathcal{E}$****.*

If $K \in \mathcal{E}$ and $X \in \mathcal{F}$ then there is a canonical equivalence

$$\mathcal{F}(K, X) \simeq XK, \quad x \mapsto x(1_K), \tag{39}$$

and we tend to identify objects and morphisms in $\mathcal{F}(K, X)$ with their images in XK.

5.2. Definition. *An object X of $\mathcal{F}$ is called a* ***category with homs internal to $\mathcal{E}$*** *when, for all $K \in \mathcal{E}$ and $x, x' \in XK$, there exists a morphism $X(x, x') \to K$ in $\mathcal{E}$ and a bijection*

$$\mathcal{E}_{/K}(L, X(x, x')) \cong (XL)\left((Xu)\,x, (XU)\,x'\right) \tag{40}$$

which is natural in $u \colon L \to K$.

5.3. Proposition. *Categories internal to $\mathcal{E}$ have homs internal to $\mathcal{E}$.*

Proof. To obtain $B(b, b')$ for a category B in $\mathcal{E}$ simply take the pullback of the morphism $(d_0, d_1) \colon B_1 \to B_0 \times B_0$ along $(b, b') \colon K \to B_0 \times B_0$. ■

5.4. Definition. *An object X of $\mathcal{F}$ is said to be* ***cocomplete at*** 1 *when the following conditions hold:*

(1) coequalizers exist in $X1$;

(2) for each $!_K \colon K \to 1$ in $\mathcal{E}$, the functor $X!_K \colon X1 \to XK$ preserves coequalizers;

(3) for objects K and L of $\mathcal{E}$, the functor $X\,\mathrm{pr}_1 \colon XK \to X(K \times L)$ has a left adjoint $\tilde{X}\,\mathrm{pr}_1$ and the canonical natural transformation $(\tilde{X}\,\mathrm{pr}_1)(X\,\mathrm{pr}_2) \to (X!_K)(\tilde{X}!_L)$ is invertible.

5.5. Theorem. *If C is a category internal to $\mathcal{E}$ and $X \in \mathcal{F}$ is cocomplete at 1 then each morphism $g \colon C \to X$ in $\mathcal{F}$ has a left extension* $\operatorname{colim} g \colon 1 \to X$ *along $!_C \colon C \to 1$ such that, for all $K \in \mathcal{E}$, the following diagram exhibits* $(\operatorname{colim} g)!_K$ *as a left extension of $g\,\mathrm{pr}_1$ along* pr_2*.*

$$\begin{tikzcd}
C \times X \arrow[rr, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner"] \arrow[drr, phantom, "="] & & K \arrow[d, "!_K"] \\
C \arrow[rr] \arrow[dr, "g"'] & \Longrightarrow & 1 \arrow[dl, "\operatorname{colim} g"] \\
& X &
\end{tikzcd}$$

Proof. We seek, in the first instance, a left adjoint to the functor $\mathcal{F}(1,X) \to \mathcal{F}(C,X)$. Recall the extension of Yoneda's Lemma given in Section 5 of [15]: namely, there is a canonical equivalence

$$\mathcal{F}(C,X) \simeq \mathcal{G}(X)C$$

where $\mathcal{G}(X) \to \mathcal{E}$ is the fibration arising on application of the Grothendieck construction to $X\colon \mathcal{E}^{\mathrm{op}} \to \mathrm{Cat}$, and $\mathcal{G}(X)C$ denotes the category of categories T in $\mathcal{G}(X)$ which lie over C and have $d_0\colon T_1 \to T_0$ cartesian (the word "full" on the second line of (5.13) in [15] should obviously be deleted!). We seek a left adjoint to the functor

$$\Delta\colon X1 \to \mathcal{G}(X)C$$

which takes z to the category in $\mathcal{G}(X)$ with object of objects $(B_0, (X!_{B_0})z)$ and each d_i cartesian. (When X has small coproducts, $\mathcal{G}(X)C$ is monadic over XC_0 and an adjoint triangle gives the result (compare (9.10) and (9.15) of [15]).

Let $\theta_i\colon (\tilde{X}!_{B_1})(Xd_i) \to \tilde{X}!_{B_0}$ be the 2-cell corresponding under adjunction to the isomorphism $(Xd_i)(X!_B) \cong X!_{B_1}$ where $d_i\colon B_1 \to B_0$. Take T in $\mathcal{G}(X)C$. The morphism $d_1\colon T_1 \to T_0$ consists of $d_1\colon B_1 \to B_0$ and a morphism $\xi\colon (Xd_0)t \to (Xd_1)t$ in XB_1, where $T_n = (B_n, t_n)$.

Let $c \in X1$ be the coequalizer of the two morphisms

$$\left(\tilde{X}!_{B_1}\right)(Xd_0)\,t_0 \xrightarrow{\theta_0 t_0} \left(\tilde{X}!_{B_0}\right)t_0 \quad \text{and}$$

$$\left(\tilde{X}!_{B_1}\right)(Xd_0)\,t_0 \xrightarrow{(\tilde{X}!_{B_1})\xi} \left(\tilde{X}!_{B_1}\right)(Xd_1)\,t_0 \xrightarrow{\theta_1 t_0} \left(\tilde{X}!_{B_0}\right)t_0.$$

A morphism $c \to z$ in $X1$ then amounts to a morphism $(B_0, t_0) \to (B_0, (X!_{B_0})z)$, which has a unique extension to a morphism $T \to \Delta(z)$ in $\mathcal{G}(X)C$.

So take $\operatorname{colim} g = c$ where g corresponds to T. Conditions (2) and (3) of "cocomplete at 1" yield the property of $(\operatorname{colim} g)!_K$ in a straightforward manner. ∎

Suppose $f\colon B \to X$ is a morphism in $\mathcal{F}$. An object $z \in X1$ is *locally isomorphic to a value of* f when there exist a regular epimorphism $R \to 1$, an object $b \in BR$, and an isomorphism

$$f_R(b) \cong (X!)\,z \tag{41}$$

in XR. Let $\mathrm{Loc}_X(f)$ denote the full subcategory of $X1$ consisting of such z.

Define the pseudofunctor

$$S\colon \mathcal{E}^{\mathrm{op}} \to \mathrm{Cat} \tag{42}$$

by $SK = \mathcal{E}_{/K}$ and

$$S(L \xrightarrow{u} K) = \mathcal{E}_{/K} \xrightarrow{u^*} \mathcal{E}_{/L}. \tag{43}$$

Recall from [15] that this is the object of $\mathcal{F}$ which gives rise to the Yoneda structure on $\mathcal{F}$ for which

$$\hat{X} = \mathcal{P}X = [X^{\mathrm{op}}, S] \tag{44}$$

and X is admissible when it has homs internal to $\mathcal{E}$, in which case X has a Yoneda morphism

$$よ_X \colon X \to \hat{X} \tag{45}$$

defined as follows: for $x \in XK$, the pseudonatural transformation

$$(よ_X K)\, x \colon \mathcal{E}(-, K) \times (X-)^{\mathrm{op}} \to S- \tag{46}$$

has component at L that functor which takes $u \colon L \to K$, $y \in XL$ to $X((Xu)x, y) \to L$ in $\mathcal{E}_{/L}$.

5.6. REMARK. For $B \in \mathcal{K}$, we have $\hat{B}K \simeq \mathcal{E}^{B^{\mathrm{op}} \times K}$. So an object of $\hat{B}1$ amounts to a discrete fibration $p \colon E \to B$. Then the discrete fibration $p \colon E \to B$ is locally isomorphic to a value of $よ_B \colon B \to \hat{B}$ precisely when it is a B-torsor. In fact, we have an equivalence of categories

$$\mathrm{Loc}_{\hat{B}}(よ_B) \simeq \mathrm{Tors}(B). \tag{47}$$

5.7. COROLLARY. *Suppose $X \in \mathcal{F}$ is cocomplete at 1 and $p \colon E \to B$ in $\mathcal{K}$ is a B-torsor. Then* $\mathrm{colim}(fp) \in X1$ *is locally isomorphic to a value of f for all $f \colon B \to X$ in $\mathcal{F}$.*

PROOF. Suppose that E is representable over R by $b \colon R \to B$. Since f is a pointwise left extension along 1_B [14], the following diagram has the left extension property.

$$\begin{tikzcd} B/b \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\xRightarrow{f\lambda}"] & R \arrow[d, "fb"] \\ B \arrow[r, "f"'] & X \end{tikzcd}$$

The span $(d_0, B/b, d_1)$ is isomorphic to $(p\,\mathrm{pr}_1, E \times R, \mathrm{pr}_2)$. It follows that fb is isomorphic to $\mathrm{colim}(fp)!_R$, and the result is proved. ∎

Consider an object X of $\mathcal{F}$. Categories constructed internally out of $\mathcal{E}$ have the form $X1$ for some such X. Objects x of XK for $K \in \mathcal{E}$ are *K-indexed families of objects of X*. Objects $z \in X1$ which are locally isomorphic to a value of $x \colon K \to X$ are defined to be *locally isomorphic to a member of the family x*. Our purpose now is to classify all such z for a given x using torsors.

Suppose X has homs internal to $\mathcal{E}$ and $x \in XK$. Form the comma object

$$\begin{tikzcd} x/x \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\Rightarrow"] & K \arrow[d, "x"] \\ K \arrow[r, "x"'] & X \end{tikzcd} \tag{48}$$

in $\mathcal{F}$. Since X is admissible and $K \in \mathcal{E}$, it follows from [15] that x/x is isomorphic to an object of $\mathcal{E}$. Thus the category

$$\begin{tikzcd} x/x \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & K \end{tikzcd} \tag{49}$$

in $\mathcal{F}$ is represented by a category $X[x]$ in $\mathcal{E}$. This category should be thought of as "consisting of the members of the family x and all the X-morphisms between them" (compare [18], [9], [19], and [20]). We obtain the factorization

$$K \to X[x] \xrightarrow{j_x} X \tag{50}$$

of $x\colon K \to X$ into a "bijective-on-objects" followed by a "fully faithful" j_x.

5.8. PROPOSITION. *In the situation of (48), (49) and (50) an object $z \in X1$ is locally isomorphic to a member of x if and only if z is locally isomorphic to a value of $j_x\colon X[x] \to X$. Thus*

$$\mathrm{Loc}_X(x) = \mathrm{Loc}_X(j_x)$$

and there is a functor

$$j_x/-\colon X1 \to \mathcal{E}^{X[x]^{\mathrm{op}}}$$

which assigns to each $z \in X1$ a discrete fibration $p\colon E \to X[x]$ in $\mathcal{K}$ for which there is diagram with the comma property in $\mathcal{F}$ as follows.

$$\begin{tikzcd} E \arrow[r, "!_E"] \arrow[d, "p"'] \arrow[dr, phantom, "\Longrightarrow"] & 1 \arrow[d, "z"] \\ X[x] \arrow[r, "j_x"'] & X \end{tikzcd}$$

PROOF. It only remains to point out that j_x/z is isomorphic to an object E of $\mathcal{K}$ by (9.7) of [15]. ■

5.9. PROPOSITION. *In the situation of Proposition 5.8, if X is cocomplete at 1 then $j_x/-$ has a left adjoint whose value at $p\colon E \to X[x]$ is $\mathrm{colim}(j_x p)$.*

PROOF. This is an immediate consequence of Theorem 5.5 and the universal property of comma objects. ■

5.10. PROPOSITION. *In the situation of Proposition 5.8, E is an $X[x]$-torsor if and only if z is locally isomorphic to a value of j_x.*

PROOF. For $b\colon R \to K$ in $\mathcal{E}$, consider the following two diagrams.

$$\begin{tikzcd} E \times R \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "="] & R \arrow[d, "!_R"] \\ E \arrow[r, "!_E"] \arrow[d, "p"'] \arrow[dr, phantom, "\Longrightarrow"] & 1 \arrow[d, "z"] \\ X[x] \arrow[r, "j_x"'] & X \end{tikzcd} \tag{51}$$

$$
\begin{tikzcd}
X[x]/b \arrow[r, "d_1"] \arrow[d] \arrow[dr, phantom, "="] & R \arrow[d, "b"] \\
X[x]^{\mathbf{2}} \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\Longrightarrow"] & X[x] \arrow[d, "j_x"] \\
X[x] \arrow[r, "j_x"'] & X
\end{tikzcd}
\tag{52}
$$

The top squares are pullbacks. The bottom squares have the comma property since j_x is fully faithful and $E \cong j_x/z$. Therefore $z!_R \cong j_x b$ if and only if $E \times R \cong X[x]/b$ as spans. ∎

5.11. THEOREM. [Classification] *Suppose $\mathcal{E}$ is finitely complete and cartesian closed. Suppose $X \in \mathcal{F}$ has homs internal to $\mathcal{E}$, is cocomplete at* 1*, and is such that the functor $X!_R \colon X1 \to XR$ is conservative for each regular epimorphism $!_R \colon R \to 1$ in $\mathcal{E}$. For all $K \in \mathcal{E}$ and $x \in XK$, the adjunction of Proposition 5.9 restricts to an equivalence of categories*

$$\mathrm{Tors}(X[x]) \simeq \mathrm{Loc}_X(x).$$

PROOF. Consider the unit $E \to j_x/\operatorname{colim}(j_x p)$ of the adjunction of Proposition 5.9 at $p \colon E \to X[x]$ in $\mathrm{Tors}(X[x])$. Since $\mathcal{E}$ is cartesian closed, pullback along a regular epimorphism $R \to 1$ is conservative. So it suffices to prove that the unit becomes invertible after applying $- \times R$ for some such R. But E becomes representable over some such R and the result then is clear.

Consider the counit $\operatorname{colim}(j_x d_0) \to z$ at $z \in \mathrm{Loc}(x)$. But z becomes isomorphic to a value of j_x on pulling back along some regular epimorphism $R \to 1$ and it is easy to see that the unit then becomes invertible. Since $X!_R$ is conservative, the result follows. ∎

We shall consider the question of existence of objects X of $\mathcal{F}$ satisfying the hypotheses of Theorem 5.11. Recall from [15] that $S \in \mathcal{F}$ as described by (42) and (43) has homs internal to $\mathcal{E}$ if and only if $\mathcal{E}$ is internally complete (see Example 2.8(1)). This is too restrictive for our purposes here so we need to consider a certain full subcategory of S.

Consider the pseudofunctor

$$P \colon \mathcal{E}^{\mathrm{op}} \to \mathrm{Cat} \tag{53}$$

whose value at K is the full subcategory of $\mathcal{E}_{/K}$ consisting of the powerful morphisms into K and whose value at $u \colon L \to K$ is pullback u^* along u. Recall the Chevalley-Beck condition as in [21] and [14].

5.12. PROPOSITION. *For any finitely complete category $\mathcal{E}$,*

(1) P has homs internal to $\mathcal{E}$;

(2) for each powerful morphism $u \colon L \to K$, the functor Pu has a left adjoint $\hat{P}u$ and satisfies the Chevalley-Beck condition on pullbacks along powerful morphisms.

If further $\mathcal{E}$ is cartesian closed, *(3) the functor $P!_R \colon P1 \to PR$ is conservative for all $R \in \mathcal{R}$.*

If further $\mathcal{E}$ has coequalizers,

(4) P is cocomplete at 1.

PROOF.

(1) If x and x' are powerful morphisms into K, take $X(x, x') \to K$ to be their internal hom (see Proposition 2.4(3)) in $\mathcal{E}_{/K}$.

(2) For $u \colon L \to K$ powerful, $\hat{P}u \colon PL \to PK$ is defined by composition with u (see Corollary 2.6(3)). The Chevalley-Beck pullback condition follows from that for S.

(3) $K \times R \to K$ is a regular epimorphism.

(4) $P1 = \mathcal{E}$ and $L \times -$ preserves coequalizers. ∎

Suppose $\mathcal{E}$ is finitely complete, cartesian closed and has coequalizers. For $K \in \mathcal{E}$, an object x of PK is a powerful morphism $x \colon M \to K$ in $\mathcal{E}$. The category $P[x]$ is the internal full subcategory of $\mathcal{E}$ determined by "the fibres of x" (see Corollary 2.19). For $B \in \mathcal{K}$, it follows from Section 5 of [15] that morphisms $f \colon B \to P$ in $\mathcal{F}$ amount to powerful discrete opfibrations $q \colon F \to B$. So $j_x \colon P[x] \to P$ gives a powerful discrete opfibration $J \to P[x]$. The equivalence of Theorem 5.11

$$\mathrm{Tors}(P[x]) \simeq \mathrm{Loc}_P(x) \tag{54}$$

and takes a torsor $E \to P[x]$ to the tensor product $E \otimes J$ of the profunctors E and J which is locally isomorphic to some fibre of x. Consequently:

5.13. PROPOSITION. *The $P[x]$-torsors classify objects of $\mathcal{E}$ locally isomorphic to fibres of x.* ∎

Theorem 4.9 can be used to show that, if B is a model in $\mathcal{E}$ for a finite limit theory richer than the theory of categories, one can deduce that $\mathrm{Tors}(B)$ is a model in Set of that theory. We shall make a lot of use of this in the next section. The following simple case however admits an easy direct proof.

5.14. PROPOSITION. *If $\mathcal{E}$ is finitely complete and cartesian closed and if B is a groupoid in $\mathcal{E}$ then* $\mathrm{Tors}(B)$ *is a groupoid.*

PROOF. For $R \in \mathcal{R}$, the functor $\mathcal{E} \to \mathcal{E}_{/R}$ is conservative. However, by Yoneda's lemma, morphisms between representable discrete fibrations over groupoids are invertible. ∎

Consider the sub-pseudofunctor

$$Q\colon \mathcal{E}^{\mathrm{op}} \to \mathrm{Cat} \tag{55}$$

of the pseudofunctor P of (53) for which the objects of each QK are the same as those of PK and the morphisms of QK are the invertible morphisms in PK. Notice that Q lands in the 2-category Gpd of groupoids.

5.15. PROPOSITION. *The object $Q \in \mathcal{F}$ satisfies all the properties stated for P in Proposition 5.12*

PROOF. The only statement which is not completely obvious is (1). This follows from the observation that in a finitely complete category $\mathcal{C}$, if exponentiation exists with respect to the two indices A and B, then one can construct an object $\mathrm{Iso}(A, B)$ (by equalizing certain pairs of morphisms out of $B^A \times A^B$ into A^A and B^B) with the property that morphisms $C \to \mathrm{Iso}(A, B)$ are in natural bijection with isomorphisms $C \times A \to C \times B$. ■

Suppose $\mathcal{E}$ is finitely complete, cartesian closed and has coequalizers. Suppose that the morphism $x\colon M \to K$ is powerful in $\mathcal{E}$. The category $Q[x]$ is the internal subcategory of $\mathcal{E}$ determined by "the fibres of x and all the isomorphisms between them"; it is a groupoid in $\mathcal{E}$. In particular, if $K = 1$ then $Q[x]$ is the group $\mathrm{Aut}(M)$ in $\mathcal{E}$ of automorphisms of M. The equivalence (54) restricts to an equivalence

$$\mathrm{Tors}(Q[x]) \simeq \mathrm{Loc}_Q(x). \tag{56}$$

As we shall see in the next section, the equivalence (54) contains quite a bit more useful information than (56).

6. Applications

6.1. FINITENESS IN A TOPOS. As a special case of Proposition 5.13, suppose $\mathcal{E}$ is an elementary topos with a natural numbers object N. Following [22], we take $x\colon N \times N \to N$ to be the composite of "addition" $N \times N \to N$ with "successor" $N \to N$. Then $P[x] = S[x]$ is the internal full subcategory E_{fin} of finite objects of $\mathcal{E}$. The fibres of x are the finite cardinals in $\mathcal{E}$. Objects of $\mathcal{E}$ locally isomorphic to the fibres of x are the Kuratowski-finite decidable objects of $\mathcal{E}$ (see [3]). Let $\mathcal{E}_{\mathrm{locfin}}$ denote the full subcategory of $\mathcal{E}$ consisting of the Kuratowski-finite decidable objects of $\mathcal{E}$; that is, $\mathcal{E}_{\mathrm{locfin}} = \mathrm{Loc}_P(x)$. Thus we have an equivalence

$$\mathrm{Tors}(E_{\mathrm{fin}}) \simeq \mathcal{E}_{\mathrm{locfin}} \tag{57}$$

which means that E_{fin}-torsors classify Kuratowski-finite decidable objects of $\mathcal{E}$.

It is well known (see [22] and [3]) that E_{fin} is an elementary topos object in $\mathcal{E}$ and that the theory of elementary toposes is a model of a finite-limit theory (see [8]). It follows from Theorem 4.9 that each $\mathrm{Tors}(R, B)$ is an elementary topos and hence from (37) that $\mathrm{Tors}(B)$ is an elementary topos (since the finite limits needed to express the structure of elementary topos commute with filtered colimits. This gives a simple conceptual proof that $\mathcal{E}_{\mathrm{locfin}}$ is an elementary topos; see [23] even includes the case where $\mathcal{E}$ has no natural numbers object.

6.2. Vector bundles. Let $\mathcal{C}$ denote the category of compactly generated spaces and let T denote a weakly separated space in $\mathcal{C}$. Put $\mathcal{E} = \mathcal{C}_{/T}$ which is a cartesian closed category by [24].

Let $\mathbb{N}$ denote the discrete space of natural numbers and let $\mathbb{R}$ denote the space of real numbers. Put

$$\mathbb{V} = \sum_{n=0}^{\infty} \mathbb{R}^n \tag{58}$$

There is a morphism $\ell \colon \mathbb{V} \twoheadrightarrow \mathbb{N}$ in $\mathcal{C}$ which is constant at n on the factor $\mathbb{R}^n$ of the coproduct.

In (54) take x to be the powerful morphism $\ell \times T \colon \mathbb{V} \times T \twoheadrightarrow \mathbb{N} \times T$ in $\mathcal{E}$. Thus we obtain an internal full subcategory $P[x] = \mathrm{Mat}_T(\mathbb{R})$ of $\mathcal{E}$ and an equivalence of categories

$$\mathrm{Tors}(\mathrm{Mat}_T(\mathbb{R})) \simeq \mathrm{Loc}_P(\ell \times T). \tag{59}$$

The category $\mathrm{Mat}_T(\mathbb{R})$ in $\mathcal{E}$ has the form $M \times T \twoheadrightarrow T$ where M is a category in $\mathcal{C}$ whose underlying category in Set is equivalent to the category of finite dimensional vector spaces over $\mathbb{R}$.

A regular epimorphism $R \twoheadrightarrow 1$ in $\mathcal{E}$ amounts to a quotient map $S \twoheadrightarrow T$ in $\mathcal{C}$. Recall that *surjective local homeomorphisms* are quotient maps and are stable under pullback. Let $\mathcal{R}'$ denote the directed ordered subset of $\mathcal{R}$ (as in (36)) consisting of surjective local homeomorphisms. We obtain, by restriction, an equivalence of categories

$$\mathrm{Tors}'(\mathrm{Mat}_T(\mathbb{R})) \simeq \mathrm{Loc}'_P(\ell \times T) \tag{60}$$

between those torsors which become representable on pulling back along some $S \twoheadrightarrow T$ in $\mathcal{R}'$ and those maps into T which become isomorphic to some fibre of $\ell \times T$ on pulling back along some $S \twoheadrightarrow T$ in $\mathcal{R}'$. Furthermore, (37) yields an isomorphism

$$\mathrm{Tors}'(\mathrm{Mat}_T(\mathbb{R})) \cong \operatorname*{colim}_{R \in \mathcal{R}'^{\mathrm{op}}} \mathrm{Tors}'(R, \mathrm{Mat}_T(\mathbb{R})). \tag{61}$$

It is an easy matter to see that $\mathrm{Loc}'_P(\ell \times T)$ *is precisely the category* $\mathrm{Vect}(T)$ *of vector bundles over* T. Putting together the above results with Theorem 4.9, we obtain the essence of some usual applications of the “clutching construction for vector bundles” and K-theory:

6.3. Proposition. *The category* $\mathrm{Vect}(T)$ *of vector bundles over* T *possesses all the structure definable by finite limits possessed by the category* $\mathrm{Mat}_T(\mathbb{R})$. *In particular, idempotents split in* $\mathrm{Vect}(T)$.

6.4. Local monadicity. Let $\mathcal{C}$ denote a finitely cocomplete, internally complete category (such as a topos). Let D denote a category in $\mathcal{C}$. Let M denote the category in $\mathcal{C}$ "consisting of the monads on D": one easily constructs this using limits as an object over the cartesian hom D^D in $\mathrm{Cat}(\mathcal{C})$. There is an action of M on D defined by composition. Let A denote the subobject of the inserter of the two morphisms

$$M \times D \underset{\mathrm{pr}_2}{\overset{\text{action}}{\rightrightarrows}} D \tag{62}$$

consisting of "the algebras for all monads on D".

In (54) $\mathcal{E}$ to be the underlying category of the 2-category $\mathrm{Cat}(\mathcal{C})$ and take x to be the fibration $A \twoheadrightarrow M$ (which is powerful by Corollary 2.17). Notice that $R \twoheadrightarrow 1$ is a regular epimorphism in $\mathcal{E}$ if and only if $R_0 \twoheadrightarrow 1$ is a regular epimorphism in $\mathcal{C}$.

The category $P[x]$ in $\mathcal{E}$ is the double category $\mathbf{Alg}(D)$ in $\mathcal{C}$ "whose objects are categories monadic over D, whose vertical morphisms are arbitrary functors, and, whose horizontal morphisms are algebraic functors". Thus we obtain an equivalence of categories

$$\mathrm{Tors}(\mathrm{Alg}(D)) \simeq \mathrm{LocAlg}(D) \tag{63}$$

between the category of torsors over $\mathrm{Alg}(D)$ and the category of categories in $\mathcal{C}$ which are locally monadic over D.

Using Corollary 2.17, we can see that $\mathrm{Alg}(D)$ is an amenable category in $\mathcal{E}$. So Theorem 4.9 applies here which, with (37), yields that all the finite-limit structure possessed by $\mathrm{Alg}(D)$ transports to $\mathrm{LocAlg}(D)$. In particular, $\mathrm{LocAlg}(D)$ becomes a double category in a natural way.

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